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Exact results in modeling planetary atmospheres—III. The general theory applied to the Earth's semi-gray atmosphere

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Abstract

We apply the semi-gray model of our previous paper to the particular case of the Earth's atmosphere, in order to illustrate quantitatively the inverse problem associated with the direct problem we dealt with before. From given climatological values of the atmosphere's spherical albedo and transmittance for visible radiation, the single-scattering albedo and the optical thickness in the visible are inferred, while the infrared optical thickness is deduced for given global average surface temperature. Eventually, temperature distributions in terms of the infrared optical depth will be shown for a terrestrial atmosphere assumed to be semi-gray and, locally, in radiative and thermodynamic equilibrium.

Keywords: Planetary atmospheres; Terrestrial atmosphere; Radiative transfer; Isotropic scattering; Semi-gray opacity; Radiative equilibrium; Temperature distribution

1. Introduction

This third part is meant as a sequel to our efforts toward an exact modeling of planetary atmospheres, begun and presented in our companion papers [1] and [2], which shall be referred to simply as I and II, respectively. Models of the kind we are envisaging go back to Emden [3], who introduced their defining feature in an attempt to explain the thermal structure of the Earth's stratosphere, discovered a dozen years before. Following the lead of Emden, we presently apply our extended model to the atmosphere of our own planet. The problem we solved in II amounts to calculating the temperatures of both the ground and the atmosphere of a planet, for a given radiative flux constant F and given data regarding two spectrally well-distinguished

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domains, the visible and the infrared: incident flux F_0, optical parameters (a, τ_b)-single scattering albedo and optical thickness- and ground reflectance r_s. The temperatures were calculated for a fixed position of the sun above the horizon, defined by the parameter μ_0, the cosine of the zenith angle of the sun, as measured with respect to the outward normal to the atmosphere.

Atmospheric modelers often face a problem different from the direct one we discussed in II. They happen to have at their ready disposal such quantities as the spherical albedo and spherical transmittance of the planet's atmosphere, together with the reflectance and the temperature of the planet's surface, whereas what they often need to determine are bulk optical parameters of the atmosphere such as (a, τ_b), as much as the temperature distribution in terms of distance from the surface. It is to this problem that we turn our attention in the present account, chiefly devoted to the terrestrial atmosphere.

We shall base our considerations on the simple but non-trivial model of a semi-gray atmosphere, neglecting the external irradiation and the ground's reflectance in the infrared domain, so as to ease the task of illustrating the inverse problem (Section 2). In Section 3, a spatial average of those quantities in II which were calculated for a fixed elevation of the sun above the horizon, will be introduced. These averages will subsequently allow us to determine the mean optical parameters of the terrestrial atmosphere, associated with either the visible or the infrared domain, by solving numerically the problem inverse to that discussed in II. We first address the calculation of a mean scattering albedo and a mean optical thickness of the atmosphere with respect to visible radiation (Section 4), and then turn to the determination of the optical thickness for infrared radiation (Section 5). The infrared scattering albedo $\bar{a}$, on the other hand, is a free parameter because, as we saw in II, the boundary fluxes do not depend upon it in our model. We will then show, for different values of this albedo, the height distributions of the global mean temperature (Section 6), while a few curves will illustrate the temperature distributions corresponding to different elevations of the sun for an infrared scattering albedo fixed at $\bar{a} = 0.2$ (Section 7).

In what follows, we shall adopt the general notation of II, so that quantities normally depending on wavelength are distinguished by a tilde (resp. a bar) when they refer to the visible (resp. infrared) domain. Equations from II will be quoted by their number, preceded by the corresponding Roman numeral II.

2. The model atmosphere

It may be in order if we briefly recall the essential feature of a semi-gray atmosphere, as one in which radiative properties differ notably according to the source of radiation, thereby imposing a distinction between at least two spectral domains. The semi-gray model introduces this least distinction, absolutely necessary if one is to escape the conclusion of an atmosphere being isothermal in radiative equilibrium, scarcely the rule in the universe. It has long been recognized that atmospheres in which convection plays an important role are almost transparent for relatively high-temperature (visible) radiation but rather opaque for cooler (infrared) radiation. A way to take into account this feature of relative transparency with respect to radiation around certain (short) wavelengths, and relative opacity with respect to radiation of different (longer) wavelengths, is to consider two spectral domains with different opacities. It was observed by Emden [3], who ignored scattering processes in his model, that for a water vapor atmosphere the absorption coefficients of both domains seemed proportional at lower altitudes, and thus he postulated a coupling of the two spectral domains. He calculated the temperature of the stratosphere as dependent on the *ratio* of both absorption coefficients, finding temperatures that could even increase with height. His model therefore does capture the main qualitative feature of planetary atmospheres with different opacities in two spectral domains.

In II, we generalized Emden's model by taking into account, in both spectral domains, isotropic scattering and reflection at the surface of the planet. The two domains were assumed to be separated by a wavelength λ_0, and radiation of smaller wavelengths was termed visible, while radiation of longer wavelengths was said to belong to the infrared domain. It should be noted that this nomenclature, developed in studies of the Earth's atmosphere, may not be always appropriate in studying other planetary atmospheres, in which both spectral domains are not so clearly severed as in the case of Earth.

In the present paper, hypotheses (i)–(vi) of Section 2 of II will be adopted, together with the additional assumptions of vanishing upper irradiation and no ground reflectivity in the infrared: $\bar{F}_0 = 0$ and $\bar{r}_s = 0$. In the visible, we adopt the value $\tilde{F}_0 = 1370\,\mathrm{W/m^2}$ for the solar constant, appropriate to an atmosphere at an astronomical unit from the Sun, and we shall choose $\tilde{r}_s = 0.15$ for the ground's reflectance, a climatological value recommended notably by Peixoto and Oort [4], Thomas and Stamnes [5] and Liou [6]. The (slightly rounded) standard contemporary value for the global and annual mean surface temperature that these authors, as well as many others, use is $T_s = 288$ K. Furthermore, the atmosphere is known to be in global radiative equilibrium, which means that its globally averaged flux constant F, the average being defined below by (2), is null.

As hypothesis (ii) of II consists in assuming that the ratio ε_e of the opacities in the visible and infrared is constant, the optical depth variables in both spectral domains become proportional:

$$\tilde{\tau} = \varepsilon_e \bar{\tau}, \quad \tilde{\tau}_b = \varepsilon_e \bar{\tau}_b. \tag{1}$$

Once we know the value of ε_e, coupling and characterizing the opacities of both radiation regimes of an atmosphere, we have the means of cataloguing planetary atmospheres according to their relative opacities in both spectral domains. In that event, a mean global temperature distribution may be calculated for a known ratio ε_e, a goal toward which we introduce a spatial averaging of geographically varying quantities.

3. Spatial averages

The quantities in II are "instantaneous" ones in that they are calculated for a fixed elevation of the sun above the horizon, described by the parameter μ_0. To every quantity $G(\mu_0)$ corresponding to a fixed position of the sun in the skies, we associate its average over the whole surface of the planet, defined by

$$G = \frac{1}{2}\int_0^1 G(\mu_0)\,\mathrm{d}\mu_0. \tag{2}$$

This global mean will be written with the same symbol as that used for the original quantity, except for the absence of the variable μ_0 from the argument list. To explain the origin of the definition (2), let us take the example of the flux $F^{\uparrow}(\tau_b, \mu_0)$ emitted by the surface when the sun is above the horizon at a zenith angle $\arccos\mu_0$. The fux emitted by the surface of the illuminated hemisphere is given by $2\pi R^2 \int_0^1 F^{\uparrow}(\tau_b, \mu_0)\,\mathrm{d}\mu_0$, where R denotes the planet's radius. The mean flux radiated by the whole surface of the planet may be defined as the ratio of the preceding flux to the total surface $4\pi R^2$ of the planet, viz. $F^{\uparrow}(\tau_b) = (1/2)\int_0^1 F^{\uparrow}(\tau_b, \mu_0)\,\mathrm{d}\mu_0$, assuredly an average flux in keeping with (2).

By proceeding in this way, we may calculate the averages of the radiative quantities introduced in II, say those of the boundary fluxes $F^{\downarrow}(0, \mu_0)$, $F^{\uparrow}(0, \mu_0)$, $F^{\downarrow}(\tau_b, \mu_0)$, $F^{\uparrow}(\tau_b, \mu_0)$, $F_s^{\uparrow}(\mu_0)$, and of the flux constant $F(\mu_0)$. We insist that for all mean values, the notation will be the same except for the missing argument μ_0. The introduction of the average of the fluxes radiated by the ground and by the atmosphere leads to the following definition of mean ground and atmospheric temperatures:

$$\sigma T_s^4 = \frac{1}{2}\int_0^1 \sigma T_s^4(\mu_0)\,\mathrm{d}\mu_0, \quad \sigma T^4(\tau) = \frac{1}{2}\int_0^1 \sigma T^4(\tau, \mu_0)\,\mathrm{d}\mu_0. \tag{3}$$

The mean quantities we have thus introduced solve the equations that result from integration over μ_0 of both members of the equations satisfied by the original quantities. From Eqs. (II, 62)–(II, 66), we obtain the following surface fluxes in the visible

$$\tilde{F}^{\downarrow}(0) = \frac{\tilde{F}_0}{4}, \quad \tilde{F}^{\uparrow}(0) = \frac{\tilde{F}_0}{4} A^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s), \tag{4}$$

$$\tilde{F}^{\downarrow}(\tilde{\tau}_b) = \frac{\tilde{F}_0}{4} T^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s), \quad \tilde{F}^{\uparrow}(\tilde{\tau}_b) = \frac{\tilde{F}_0}{4} \tilde{r}_s T^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s). \tag{5}$$

In these equations, the spherical albedo A^* and the spherical transmittance T^* are given explicitly by Eqs. (A7)–(A8) in Appendix A of II.

The infrared boundary fluxes may be calculated by integrating Eqs. (II, 84), (II, 89)–(II, 92) over μ_0, where by our additional hypotheses in Section 2 we now have to set $\bar{F}_0 = 0$, $\bar{r}_s = 0$ and $\bar{F} = 0$, yielding

$$\bar{F}^{\downarrow}(0) = 0, \tag{6}$$

$$\bar{F}^{\uparrow}(0) = [1 - A^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s)]\frac{\tilde{F}_0}{4}, \tag{7}$$

$$T_{11}(1, \bar{\tau}_b)\bar{F}^{\uparrow}(\bar{\tau}_b) = (1 - \tilde{r}_s)T^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s)\frac{\tilde{F}_0}{4} + \bar{F}^{\downarrow}_{\text{atm}}(\bar{\tau}_b), \tag{8}$$

$$T_{11}(1, \bar{\tau}_b)\bar{F}^{\downarrow}(\bar{\tau}_b) = R_{11}(1, \bar{\tau}_b)(1 - \tilde{r}_s)T^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s)\frac{\tilde{F}_0}{4} + \bar{F}^{\downarrow}_{\text{atm}}(\bar{\tau}_b), \tag{9}$$

$$T_{11}(1, \bar{\tau}_b)\bar{F}^{\uparrow}_s = (1 - \tilde{r}_s)T^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s)\frac{\tilde{F}_0}{4} + \bar{F}^{\downarrow}_{\text{atm}}(\bar{\tau}_b). \tag{10}$$

On the left-hand side of (10) we replaced $T^*(1, \bar{\tau}_b, \bar{r}_s = 0)$ by $T_{11}(1, \bar{\tau}_b)$, in view of Equation (II, A8).

By integrating over μ_0 both members of Eq. (II, 78), we derive the following average infrared flux through the lower boundary plane as the result of absorption of visible light by the atmosphere:

$$\bar{F}^{\downarrow}_{\text{atm}}(\bar{\tau}_b) = \frac{1}{2}\int_0^1 \bar{F}^{\downarrow}_{\text{atm}}(\bar{\tau}_b, \mu_0)\,\mathrm{d}\mu_0, \tag{11}$$

$$= (1 - \tilde{a})\frac{\tilde{F}_0}{4}[(B_0|\hat{B}_0)(\tilde{a}, \tilde{\tau}_b, \varepsilon_e) + \tilde{r}_s T^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s)(B_0|B_0)(\tilde{a}, \tilde{\tau}_b, \varepsilon_e)]. \tag{12}$$

The coefficients $(B_0|B_0)$ and $(B_0|\hat{B}_0)$ were defined by Eqs. (II, 75) and (II, 76), respectively.

The thermal flux radiated by the ground is connected to the average temperature of the perfectly emitting ground ($\bar{e}_s = 1$) by relation (II, 93), which integrated over μ_0 gives

$$\bar{F}^{\uparrow}_s \sim \sigma T_s^4. \tag{13}$$

The average temperature of the ground is inferred from the model parameters by means of Eqs. (10), (12) and (13), combining to produce

$$T_{11}(1, \bar{\tau}_b)\sigma T_s^4 = \frac{\tilde{F}_0}{4}\{(1 - \tilde{r}_s)T^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s) + (1 - \tilde{a})[(B_0|\hat{B}_0)(\tilde{a}, \tilde{\tau}_b, \varepsilon_e) + \tilde{r}_s T^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s)(B_0|B_0)(\tilde{a}, \tilde{\tau}_b, \varepsilon_e)]\}. \tag{14}$$

As to the average temperature of the atmosphere we get, upon integrating (II, 101) over μ_0, the expression

$$\sigma T^4(\bar{\tau}) = \frac{1}{2}\sigma T_s^4 B_0(1, \bar{\tau}_b, \bar{\tau}_b - \bar{\tau}) + \varepsilon_a\frac{\tilde{F}_0}{8}[c_0(\tilde{a}, \tilde{\tau}_b, \bar{a}, \bar{\tau}_b, \bar{\tau}) + \tilde{r}_s T^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s)c_0(\tilde{a}, \tilde{\tau}_b, \bar{a}, \bar{\tau}_b, \bar{\tau}_b - \bar{\tau})]. \tag{15}$$

The right-hand side involves the ratio ε_a of the absorption coefficients in the visible and the infrared, given by (II, 95), as well as the function c_0 specified by Eq. (II, 103). In the first term of the right-hand side, $\bar{F}^{\uparrow}(\bar{\tau}_b)$ has been replaced by $\bar{F}^{\uparrow}_s \sim \sigma T_s^4$, in accordance with Eq. (II, 85), averaged over μ_0 for $\bar{r}_s = 0$. Let us recall the meaning of each term on the right-hand side of (15): the first term describes heating of the atmosphere by infrared radiation emitted by the ground, while the terms within square brackets represent heating due to conversion of visible radiation, either reaching the top of the atmosphere from above (first term in brackets) or having been reflected by the planet's surface (second term).

4. Mean albedo and mean optical thickness of the Earth's atmosphere in the visible domain

The mean scattering albedo $\tilde{a}$ and the optical thickness $\tilde{\tau}_b$ of the terrestrial atmosphere in the visible can be deduced from given values $\tilde{A}^*$ and $\tilde{T}^*$ of its spherical albedo and transmittance by solving the system

$$A^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s) = \tilde{A}^*, \quad T^*(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s) = \tilde{T}^*. \tag{16}$$

We have adopted the climatological values $\tilde{A}^* = 0.3$ and $\tilde{T}^* = 0.55$. The former is the widely accepted global mean, and is consonant with an effective temperature $T_{\rm ef} = 255\,{\rm K}$ and a solar constant $\tilde{F}_0 = 1370\,{\rm W/m^2}$. As to the value of $\tilde{T}^*$, we deduced it from the heat balance of Earth's atmosphere, summarized for instance by Thomas and Stamnes [5], or Liou [6]. The values derived from these recent references coincide. Entering the expressions (II, A7) and (II, A8) for $A^*(\tilde{a}, \tilde{\tau}_{\rm b}, \tilde{r}_{\rm s})$ and $T^*(\tilde{a}, \tilde{\tau}_{\rm b}, \tilde{r}_{\rm s})$ in the left-hand side of Eqs. (16), one obtains a system of two equations for the coefficients R_{11} and T_{11}, whose solution is

$$R_{11}(\tilde{a}, \tilde{\tau}_{\rm b}) = \frac{\tilde{A}^* - \tilde{r}_{\rm s}(\tilde{T}^*)^2}{1 - \tilde{r}_{\rm s}^2(\tilde{T}^*)^2}, \quad T_{11}(\tilde{a}, \tilde{\tau}_{\rm b}) = \frac{(1 - \tilde{r}_{\rm s}\tilde{A}^*)\tilde{T}^*}{1 - \tilde{r}_{\rm s}^2(\tilde{T}^*)^2}. \tag{17}$$

With $\tilde{A}^* = 0.3$, $\tilde{T}^* = 0.55$ and $\tilde{r}_{\rm s} = 0.15$, we find $R_{11}(\tilde{a}, \tilde{\tau}_{\rm b}) = 0.25637$ and $T_{11}(\tilde{a}, \tilde{\tau}_{\rm b}) = 0.52885$. With the expressions (II, 57) and (II, 58) for the bimoments R_{11} and T_{11}, a system of two equations for the parameters $\tilde{a}$ and $\tilde{\tau}_{\rm b}$ is arrived at, which can be solved numerically. To this end, tables for the moments of order 0 and 1 of the X- and Y-functions are needed, which fortunately are available in the literature: see [7, Section 9.6]. But we actually used tables set up with our own transfer code ARTY, obtaining for the parameters in question $\tilde{a} = 0.8198$ and $\tilde{\tau}_{\rm b} = 0.7176$.

Our value for the mean scattering albedo is lower than those given by other authors, Eschelbach [8] among them, who find values close to 0.9. The difference may partly be due to the fact that light scattering in the visible is, for simplicity, supposed to take place isotropically in our calculations, whereas it is not so restricted in [8]: cf the similarity relations of van de Hulst, whose expression based on the two-stream approximation can be found in [9, p. 361]. Note also that we used a relatively low value of the spherical transmittance $\tilde{T}^*$ (0.55), certainly reflecting the average effect of the mean partial cloud cover of the sky (cf [6, p. 466]), while this aspect was ignored in [8]. For a higher value of $\tilde{T}^*$, say 0.70, we would have found a higher value of the albedo $\tilde{a}$ (0.89) and a smaller optical thickness $\tilde{\tau}_{\rm b}$ (0.45). However, we may safely posit that the discrepancy is mainly due to the fact that scattering in our model is isotropic, in contrast with what happens in a cloudy and hazy atmosphere.

5. Optical thickness of the Earth's atmosphere in the infrared domain

We determined this parameter by solving Eq. (14). Of course, we took care to choose the values $T_{\rm s} = 288\,{\rm K}$, $T^*(\tilde{a}, \tilde{\tau}_{\rm b}, \tilde{r}_{\rm s}) = 0.55$, $\tilde{r}_{\rm s} = 0.15$ and $\tilde{F}_0 = 1370\,{\rm W/m^2}$. The coefficients $(B_0|B_0)$ and $(B_0|\hat{B}_0)$ were calculated from their definitions (II, 75) and (II, 76). We based the calculations on accurate tables for the function B_0, reckoned with our code ARTY from the exact expression for the function $\xi_0 = (a/2)B_0$ given in [10]. We checked our calculation by means of Eq. (II, 81), which upon inserting the adopted values for $\tilde{a}$, R_{11} and T_{11} presently reads $(B_0|B_0) + (B_0|\hat{B}_0) = 1.19190$. Finally, if we replace $\bar{\tau}_{\rm b}$ by $\tilde{\tau}_{\rm b}/\varepsilon_{\rm e}$ in $T_{11}(1, \bar{\tau}_{\rm b})$, with $\tilde{\tau}_{\rm b} = 0.7176$, Eq. (14) becomes an equation in $\varepsilon_{\rm e}$, which can be solved numerically. The value thus obtained, namely $\varepsilon_{\rm e} = 0.58508$, yields for the infrared optical thickness $\bar{\tau}_{\rm b} = 1.2265$.

This example makes it quite clear why the introduction of global averages for the main quantities of the atmosphere is of such interest. Identifying the average value (3) of the ground's temperature with the value given in the literature, we wind up facing the simpler problem of dealing with Eq. (14), far more tractable than the problem of dealing with (18) for a fixed position of the sun, to be considered below.

6. Calculation of the average temperature of the Earth's atmosphere

We determine the temperature of the atmosphere with the help of relation (15). In order to evaluate the function c_0 from its definition (II, 103), it is necessary to calculate, in the first place, the function B_0 by solving the integral equation (II, 34), proceeding then to calculate the function b_0 by solving Eq. (II, 100) for $\bar{\tau}_{\rm b} = 1.2265$, $\bar{\tau} = \tilde{\tau}/\varepsilon_{\rm e}$ and $\varepsilon_{\rm e} = 0.58508$. As already mentioned, the function B_0 was determined with our code ARTY from its exact expression. As to the integral equation (II, 100), we solved it numerically by two different methods: the method ALI (Accelerated Lambda Iteration) described for example in [11], and the method FRA (Finite Rank Approximation) described in [12]. We have already pointed out that the albedo $\bar{a}$ in the infrared is a free parameter of our model, because we ignored the atmosphere's own emission in the visible domain [hypothesis (iv) of Section 2 of II]. Indeed, as we saw, this assumption entails that the infrared fluxes radiated across the two boundary planes become independent of the albedo $\bar{a}$. This remark applies in particular to the temperature of the ground, which is not coupled to the albedo $\bar{a}$.

Table 1 and Fig. 1 give values of $T(\bar{\tau})$ corresponding to $\bar{a} = 0$, 0.4 and 0.8. Note that $T(\bar{\tau})$ increases with $\bar{\tau}$, but much less so when $\bar{a} > 0.5$. In I, we have shown that the average temperature of a gray atmosphere in global radiative equilibrium is constant.

Table 1
Values of $T(\bar{\tau})$ and T_s for $\bar{a} = 0, 0.4$ and 0.8

$\bar{\tau}$	$T(\bar{\tau})$			$\bar{\tau}$	$T(\bar{\tau})$		
	$\bar{a} = 0$	0.4	0.8		$\bar{a} = 0$	0.4	0.8
0	220.08	226.51	252.49	0.6623	258.86	261.42	273.24
0.0368	224.91	230.73	254.73	0.6991	260.16	262.63	274.04
0.0736	228.42	233.82	256.42	0.7359	261.42	263.80	274.82
0.1104	231.43	236.49	257.91	0.7727	262.65	264.94	275.59
0.1472	234.13	238.90	259.28	0.8095	263.85	266.06	276.34
0.1840	236.59	241.10	260.56	0.8463	265.03	267.15	277.08
0.2208	238.88	243.16	261.77	0.8831	266.18	268.23	277.81
0.2576	241.01	245.09	262.92	0.9199	267.31	269.28	278.54
0.2944	243.02	246.91	264.02	0.9567	268.42	270.32	279.25
0.3312	244.93	248.64	265.08	0.9935	269.52	271.35	279.96
0.3680	246.74	250.29	266.10	1.0303	270.61	272.36	280.67
0.4047	248.47	251.87	267.08	1.0671	271.69	273.38	281.37
0.4415	250.12	253.38	268.04	1.1039	272.78	274.40	282.08
0.4783	251.71	254.84	268.96	1.1406	273.88	275.43	282.81
0.5151	253.24	256.24	269.86	1.1774	275.02	276.50	283.56
0.5519	254.72	257.60	270.74	1.2142	276.27	277.67	284.40
0.5887	256.14	258.91	271.59	1.2265	276.79	278.17	284.76
0.6255	257.52	260.19	272.42	T_s	288.00	288.00	288.00

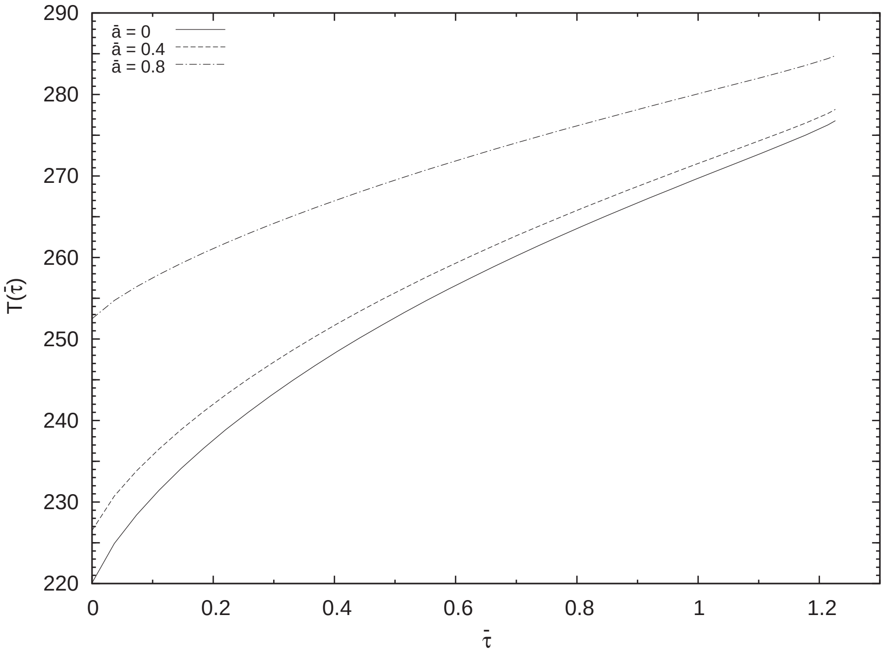

Fig. 1. $T(\bar{\tau})$ versus $\bar{\tau}$ for $\bar{a} = 0$ (solid line), 0.4 (dashed line) and 0.8 (dot–dashed line).

7. Calculation of temperatures for various fixed positions of the sun

In order to test the sensitivity of the temperatures of the ground and the atmosphere as the elevation of the sun above the horizon is varied, we calculated these temperatures for four values of μ_0 and a given infrared albedo ($\bar{a} = 0.2$): $\mu_0 = 0.1,\ 0.25,\ 0.5$ and 1, all based on Eqs. (II, 94) and (II, 101). For $F = 0$, $\bar{F}_0 = 0$, $\bar{r}_s = 0$, $\bar{e}_s = 1$ and $\bar{F}^{\uparrow}(\bar{\tau}_b, \mu_0) = \bar{F}_s^{\uparrow}(\mu_0) = \sigma T_s^4(\mu_0)$, these formulas reduce to

$$T_{11}(1, \bar{\tau}_b)\sigma T_s^4(\mu_0) = \tilde{F}_0\{(1 - \tilde{r}_s)\mu_0 T(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s, \mu_0) + (1 - \tilde{a})[(B|\hat{B}_0)(\tilde{a}, \tilde{\tau}_b, \varepsilon_e, \mu_0) + \mu_0 \tilde{r}_s T(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s, \mu_0)(B_0|B_0)(\tilde{a}, \tilde{\tau}_b, \varepsilon_e)]\}. \tag{18}$$

$$\sigma T^4(\bar{\tau}, \mu_0) = \frac{1}{2}\sigma T_s^4(\mu_0) B_0(1, \bar{\tau}_b, \bar{\tau}_b - \bar{\tau}) + \varepsilon_a \frac{\tilde{F}_0}{4}[c(\tilde{a}, \tilde{\tau}_b, \bar{a}, \bar{\tau}_b, \bar{\tau}, \mu_0) + 2\tilde{r}_s \mu_0 T(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s, \mu_0) c_0(\tilde{a}, \tilde{\tau}_b, \bar{a}, \bar{\tau}_b, \bar{\tau}_b - \bar{\tau})]. \tag{19}$$

The atmospheric flux transmittance $T(\tilde{a}, \tilde{\tau}_b, \tilde{r}_s, \mu_0)$ is given by relation (II, A2), and the coefficients $(B|\hat{B}_0)$ and $(B_0|B_0)$ in (18) were calculated from their definitions (II, 74) and (II, 75). The function B was evaluated by running our code ARTY, starting with the (yet unpublished) exact solution of the integral equation (II, 29). Eq. (19) involves the ratio ε_a of the absorption coefficients in the visible and the infrared, expressed by (II, 95), as well as the functions c and c_0, defined by Eqs. (II, 102) and (II, 103), respectively. c_0 is calculated as described in Section 6; to compute the values of the function c by means of its definition (II, 102), we first have to determine the function b by solving the integral equation (II, 98) numerically (taking advantage of the methods ALI [11] and FRA [12] applied conjointly). The results are exhibited in Table 2 and illustrated in Fig. 2.

So as to compare the temperature distribution of a gray atmosphere with that of a semi-gray atmosphere, subject to the same external irradiation, we collate the temperature curves corresponding to the sun's position $\mu_0 = 0.25$ in both the gray model (curve corresponding to $\tau_b = 1$ of Fig. 1 in I) and the present one ($\tilde{\tau}_b = 0.7176$, $\bar{\tau}_b = 1.2265$). The temperature of the gray atmosphere decreases from 311 to 256 K when the ground is approached, whereas the semi-gray model is almost isothermal, with temperatures close to 245 K. This feature may be explained as follows: in the present model, visible self-radiation being suppressed, the atmosphere is heated by the visible radiation it absorbs and is cooled at the same time by the infrared radiation

Table 2
Values of $T(\bar{\tau}, \mu_0)$ and $T_s(\mu_0)$ for $\bar{a} = 0.2$ and $\mu_0 = 0.1, 0.25, 0.5$ and 1

$\bar{\tau}$	$T(\bar{\tau}, \mu_0)$				$\bar{\tau}$	$T(\bar{\tau}, \mu_0)$			
	$\mu_0 = 0.1$	0.25	0.5	1		$\mu_0 = 0.1$	0.25	0.5	1
0	197.66	227.15	262.86	310.99	0.6623	184.61	247.02	305.97	374.38
0.0368	195.90	230.74	268.79	318.81	0.6991	184.98	247.65	307.31	376.52
0.0736	193.49	232.85	272.93	324.52	0.7359	185.36	248.27	308.60	378.59
0.1104	191.26	234.48	276.41	329.44	0.7727	185.76	248.89	309.86	380.62
0.1472	189.33	235.84	279.47	333.84	0.8095	186.18	249.50	311.09	382.59
0.1840	187.72	237.02	282.24	337.87	0.8463	186.61	250.10	312.28	384.52
0.2208	186.42	238.07	284.77	341.61	0.8831	187.05	250.71	313.45	386.42
0.2576	185.40	239.02	287.11	345.11	0.9199	187.50	251.31	314.59	388.27
0.2944	184.63	239.91	289.30	348.40	0.9567	187.95	251.92	315.72	390.10
0.3312	184.08	240.74	291.35	351.52	0.9935	188.41	252.52	316.83	391.90
0.3680	183.71	241.53	293.29	354.49	1.0303	188.88	253.13	317.93	393.69
0.4047	183.50	242.29	295.13	357.33	1.0671	189.35	253.74	319.02	395.47
0.4415	183.41	243.02	296.88	360.04	1.1039	189.83	254.36	320.11	397.25
0.4783	183.44	243.72	298.55	362.65	1.1406	190.33	255.01	321.22	399.05
0.5151	183.55	244.41	300.15	365.16	1.1774	190.85	255.68	322.36	400.91
0.5519	183.74	245.08	301.69	367.58	1.2142	191.43	256.44	323.62	402.96
0.5887	183.99	245.74	303.17	369.92	1.2265	191.67	256.76	324.14	403.82
0.6255	184.28	246.38	304.60	372.19	$T_s(\mu_0)$	197.04	263.94	335.29	421.41

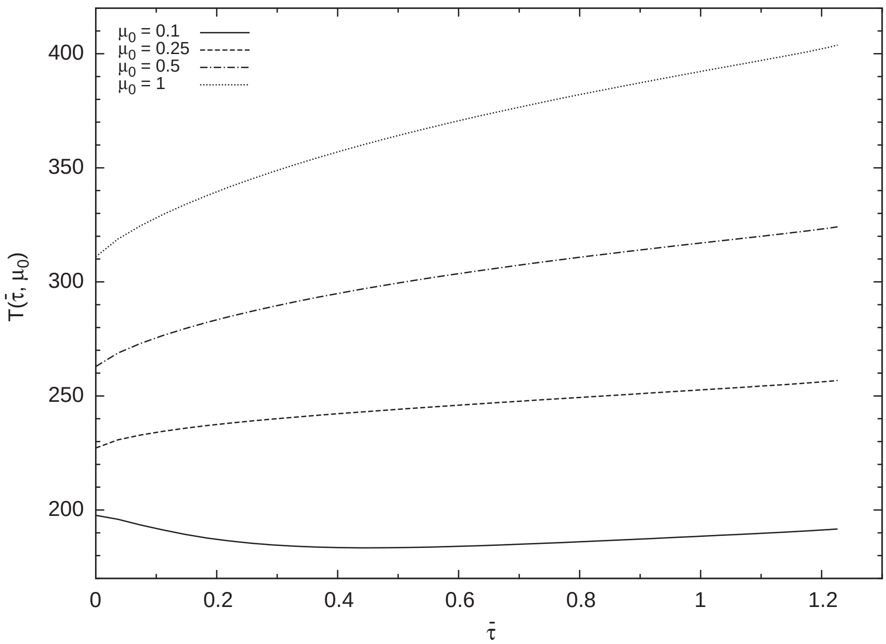

Fig. 2. $T(\bar{\tau}, \mu_0)$ versus $\bar{\tau}$ for $\bar{a} = 0.2$ and $\mu_0 = 0.1$ (solid line), 0.25 (dashed line), 0.5 (dot-dashed line) and 1 (dotted line).

it emits. Since $\tilde{k}_e = \varepsilon_e \bar{k}_e$ with $\varepsilon_e = 0.585 < 1$, the semi-gray atmosphere is more transparent in the visible part of the spectrum than the gray atmosphere, for which invariably $\tilde{k}_e = \bar{k}_e$. It must thus be less hot than the latter, and even more so close to the upper layers of the atmosphere where the radiative field is dominated by the visible radiation of the external source.

The findings in this section have important bearings on the interpretation of the dynamics of planetary atmospheres that are similar to the Earth's atmosphere. Atmospheres more transparent in the visible domain than in the infrared allow more radiation to reach the surface, while less radiation escapes to space, keeping the surface warm and thereby destabilizing the lower layers of the atmosphere, which in turn gives rise to non-radiative processes like convection that will straighten the strong gradients set up by radiative equilibrium near the surface. As soon as this happens, the solutions here obtained are no longer stable. Only for atmospheres less transparent in the visible can we expect the radiative equilibrium solutions to be stable with respect to convective perturbations.

8. Conclusions

We began by computing the main radiative quantities of an atmosphere illuminated by a sun at a fixed elevation above the horizon, and then introduced a natural averaging with respect to the sun's position μ_0, amounting to a global average over the planet's surface. The latter average served us well in this paper devoted to the Earth's atmosphere. In effect, we were able, by identifying the global average of the surface temperature with its climatological value of 288 K, to deduce a "climatological" optical thickness of the atmosphere with respect to infrared radiation, thus showing the worth of the inverse problem we set out to illustrate.

The present model does not address the spectral complexities of radiative transfer with any realism other than by taking account of the generally differing transparencies of a planetary atmosphere for visible and infrared radiation. However, it is simple enough to lend itself to an *entirely analytical* treatment of its foundation. It constitutes a first step toward the truly non-gray character of real planetary atmospheres and provides, as Goody and Yung [9] have stated with full justice, "an irreplaceable first step in a number of fields: the atmospheres of other planets, stellar atmospheres, the earth's primitive atmosphere". We regard this model as setting the stage for analytical or semi-analytical models of increasing complexity, by successively relaxing the restrictive hypotheses, as we had already pointed out in our conclusions drawn from II, where some hints for such generalizing have been given. Moreover, there is, at the time of writing, no alternative to a semi-gray model—with its unavoidable concomitant assumptions—in computer-intensive studies of climatological issues. In the latter, the numerical calculations of the radiative transfer module need to be reduced to an absolute minimum, if we are to understand better the long-term effects of changes in radiative transfer through changes of optical parameters of our atmosphere. Nor is a simpler exact route available to modeling the long-term evolution of our Earth's atmosphere than by avoiding the clutter of spectrally-resolved (non-gray) models, and investigations of past climates involving the whole of the system's dynamics can be expected to improve, if bulk formulae for radiative effects are replaced with a semi-gray model of the radiation processes that take place in the atmosphere.

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