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A Class of Viscosity Profiles for Oil Displacement in Porous Media or Hele-Shaw Cell

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Abstract. This paper is devoted to the study of the stability of oil displacement in porous media. Results are applied to the secondary oil recovery process: the oil contained in a porous medium is obtained by pushing it with a second fluid (usually water). As in Saffman and Taylor (1958) and Gorell and Homsy (1983) the porous medium will be modelized by a Hele-Shaw cell. If the second fluid is less viscous, the fingering phenomenon appears, first studied by Saffman and Taylor (1959). In order to minimize this instability, we consider, as in Gorell and Homsy (1983), an intermediate polymer-solute region (i.r.), with a variable viscosity μ, between water and oil. This viscosity increases from water to oil. The linear stability of the interfaces is governed by a Sturm–Liouville problem which contains eigenvalues in the boundary conditions. Its characteristic values are the growth constants of the perturbations. The stability can be improved by choosing a "minimizing" viscosity profile μ which gives us an *arbitrary small positive* growth constant. In this paper, we suggest a *class of minimizing* profiles. This main result is obtained by considering the Rayleigh quotient to estimate – *without any discretization* – the characteristic values of the above Sturm–Liouville problem. A finite-difference procedure and Gerschgorin's localization theorem were used by Carasso and Paşa (1998) to solve the above problem. *A formula* of an *exponential* viscosity profile in (i.r.) was obtained. The new class of minimizing viscosity profiles described in this paper includes *linear* and *exponential* profiles. The corresponding total amount of polymer and the (i.r.) length are estimated in terms of the limit value of μ on the (i.r.) – oil interface. Our results are compared with the previous theoretical viscosity profiles. We show that the linear case is more favorable compared with the exponential profile. We give lower estimates of the total amount of polymer and of the (i.r.) length for a *given improvement of the stability* , compared with the Saffman–Taylor case.

Key words: Rayleigh quotient, Sturm–Liouville eigenproblem, Hele–Shaw model, oil recovery, flow in porous media.

1. Introduction

This paper is devoted to the study of the stability of oil displacement in porous media. Results are applied to the secondary oil recovery process: the oil contained in a porous medium is obtained by pushing it with a second immiscible fluid (usually water). As in Saffman and Taylor (1958) and Gorell and Homsy (1983) the porous medium will be modelized by a Hele–Shaw cell and then an interface exists between the two immiscible fluids. If the second fluid is less viscous, the fingering phenomenon appears, first studied by Saffman and Taylor (1959) and Chouke *et al.* (1959). In order to minimize this instability, we consider, as in Gorell and Homsy (1983), an intermediate region (i.r.) containing a polymer-solute with a variable viscosity μ, between water and oil. Indeed, the surface tension on the interface can improve stability.

The porous medium is saturated by *three immiscible fluids*: water, polymer-solute and oil, separated by two interfaces. A surface tension has been considered on the water – (i.r.) and on the (i.r.) – oil interfaces (see Figure 1).

The unknown viscosity μ in (i.r.) is a parameter which will be used to improve the interfaces stability. We suppose that μ is a linear invertible function of the polymer concentration and that it increases from water to oil. As in Gorell and Homsy (1983) we consider that the viscosity is continuous on the interface water – (i.r.). Therefore, the viscosity is discontinuous on the (i.r.) – oil interface only and then a surface tension acts on this interface only.

The three regions are moving due to the water velocity U far upstream. The flow is given by

- Darcy law;
- the continuity equation of the velocity;
- a "conservation" law of the viscosity μ.

On the two interfaces, a simplified Laplace's law is used to describe the contact conditions between the immiscible fluids (see Appendix B). As in Gorell and Homsy (1983), we consider a steady basic solution with straight initial interfaces.

The interface stability is governed by a Sturm–Liouville problem, containing eigenvalues in the boundary conditions. The characteristic values of this problem are the growth constants (in time) of the perturbations. The stability of the interfaces is improved when the maximum characteristic value in the above problem is smaller than in the Saffman–Taylor case.

Before describing the main result of this paper, we give some previous results on this topic:

- Gorell and Homsy (1983) obtained a numerical exponential viscosity profile, which give an improved stability, according to previous experimental results of Mungan (1971), Pearson (1977), Shah and Schecter (1977) and Uzoigwe *et al.* (1974);
- An asymptotic analysis of the above model is given by Paşa and Polisevski (1992) in the case of a small quantity of polymer;
- An existence theorem for a minimizing viscosity in (i.r.) has been obtained by Paşa (1996) by using the Rayleigh quotient;

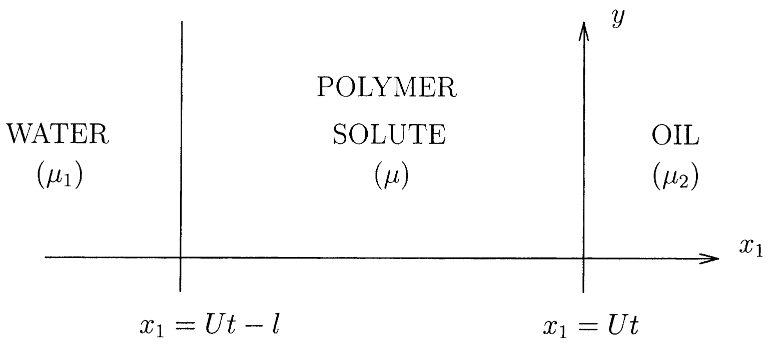

Figure 1. The three regions of the porous medium.

- An explicit formula for a minimizing viscosity in (i.r.) has been obtained by Carasso and Paşa (1998). The above Sturm–Liouville problem was discretized by the finite-difference method. Gerschgorin's localization theorem was used to estimate the characteristic values. The obtained viscosity profile in (i.r.) was *exponential*, accordingly to the numerical results of Gorell and Homsy (1983);
- A convergence theorem has been proved by Carasso and Paşa (2000) to justify the previous discretization;
- A "very slow" viscosity profile in (i.r.) was obtained in Paşa (2002) by using the result of Carasso and Paşa (1998). This "very slow" *exponential* profile gives us a growth constant which is similar to the corresponding Saffman–Taylor value: the water viscosity was replaced by the limit value of the viscosity on the (i.r.) – oil interface.

The present paper is divided in three parts. Section 2 is devoted to the description of the stability problem. In Section 3, which includes the main result of the paper, we consider the Rayleigh quotient to estimate – *without any discretization* – the characteristic values of the considered Sturm–Liouville problem. Finally, Section 4 is devoted to describe a *class* of minimizing viscosity in (i.r.), including *linear* and *sub-exponential* profiles. This viscosities give us an *arbitrary small positive* growth constant. Comparisons are made between our result and previous theoretical and numerical results – cited above – and between linear and sub-exponential profiles. We remark that the linear profile is more favorable: a smaller amount of polymer and a smaller (i.r.) length are necessary to get the same growth constant. Finally we compute the necessary (i.r.) length and the corresponding amount of polymer to get a given improvement of stability compared to the Saffman–Taylor case.

2. Review of the Stability Problem

We study here the model introduced by Gorell and Homsy (1983). Numerical results concerning this model were obtained in Daripa *et al.* (1986, 1988a, 1988b) and Daripa (1987). The model was studied also in Paşa and Polisevski (1992), Paşa (1996, 2002) and Carasso and Paşa (1998, 2000).

A homogeneous porous medium is considered in the plane $x_1 0 y$ and the Hele-Shaw approximation is used. The medium is saturated with three immiscible fluids: water (with the constant viscosity μ_1), polymer (with the unknown variable viscosity μ) and oil (with the constant viscosity μ_2). The three regions are moving due to the water velocity U at infinity upstream in the positive Ox_1 direction. The polymer is contained in the intermediate region (i.r.) (see Figure 1).

We have two sharp interfaces: water – (i.r.) and (i.r.) – oil, where we consider a simplified expression of Laplace's law: the pressure drop is balanced by the curvature times the surface tension; moreover the normal component of the velocity is continuous (see Appendix B).

In the intermediate region, we expect that the viscosity μ is an increasing function of the distance.

In the three regions of the porous medium, the flow is given by the continuity equation for the velocity and the Darcy law.

As in the paper of Gorell and Homsy (1983), *adsorption, dispersion and diffusion of the polymer-solute in the porous medium are neglected*. Then the concentration $c(x_1, t)$ verifies

$$Dc/Dt = 0.$$

We assume also that the relationship between polymer-solute concentration and viscosity, $\mu(c)$, is known and that is linear and invertible. We have $Dc/Dt = (dc/d\mu)D\mu/dt$, then we get $D\mu/dt = 0$.

Therefore, the flow is governed by the following system:

$$\partial u/\partial x_1 + \partial v/\partial y = 0, x_1 \in \mathbb{R},\ x_1 \notin \text{interfaces},\ y \in \mathbb{R}, \tag{1}$$

$$\partial P/\partial x_1 = -\delta\mu u, x_1 \in \mathbb{R},\ x_1 \notin \text{interfaces},\ y \in \mathbb{R}, \tag{2}$$

$$\partial P/\partial y = -\delta\mu v, x_1 \in \mathbb{R},\ x_1 \notin \text{interfaces},\ y \in \mathbb{R}, \tag{3}$$

$$\partial \mu/\partial t + u\partial\mu/\partial x_1 + v\partial\mu/\partial y = 0, x_1 \in \text{(i.r.)},\ y \in \mathbb{R}, \tag{4}$$

where (u, v) is the velocity of the fluid, P is the pressure and δ is the *constant* permeability of the medium. The above equations admit the following basic solution describing the steady displacement:

$$u = U, \qquad v = 0, \qquad \mu = \delta\mu_b(x_1 - Ut), \qquad P = -U\int \delta\mu_b(s - Ut)\mathrm{d}s, \tag{5}$$

where μ_b is an arbitrary function.

Then we introduce the moving reference system $x = x_1 - Ut$. The above basic solution let us consider an intermediate region (i.r.) with constant length l at the left of the origin in the moving coordinates xOy. We emphasize that μ_b verifies the following properties:

$$\mu_b \in C^\infty(i.r.), \qquad \mu_1 \leq \mu_b(x) < \mu_2, \qquad \mu_b'(x) > 0, \qquad x \in \text{(i.r.)}. \quad (6)$$

We have two sharp straight interfaces in the moving reference xOy: the water – (i.r.) interface, which corresponds to $x = -l$ and the (i.r.) – oil interface, which corresponds to $x = 0$. On the interfaces we consider the above simplified Laplace's law (see Appendix B).

We study the linear stability of the interfaces defined above. We consider the small perturbations (u', v', P', μ') and then we get the following system:

$$\partial u/\partial x + \partial v/\partial y = 0, x \in \mathbb{R},\ x \notin \{-l, 0\},\ y \in \mathbb{R}, \quad (7)$$

$$\partial P'/\partial x = -\mu' U - \mu_b u', x \in \mathbb{R},\ x \notin \{-l, 0\},\ y \in \mathbb{R}, \quad (8)$$

$$\partial P'/\partial y = -\mu_b v', x \in \mathbb{R},\ x \notin \{-l, 0\},\ y \in \mathbb{R}, \quad (9)$$

$$\partial \mu'/\partial t + u' \cdot \mathrm{d}\mu_b/\mathrm{d}x = 0, x \in (-l, 0),\ y \in \mathbb{R}. \quad (10)$$

The last relation is proved in Gorell and Homsy (1983) and used in Paşa (2002). As the problem (7)–(10) is linear, we can decompose the perturbations in Fourier components. First, let consider the horizontal component of the velocity perturbation:

$$u'(x, y, t) = f(x)\ \exp(\mathrm{i}ky + \sigma t), \quad (11)$$

where $f(x)$ is the amplitude of the perturbation, k is the wavenumber in the Oy direction and σ is the growth constant in time. The corresponding expressions for v', P' and μ' are given by applying (7), (8) and (10). As in Gorell and Homsy (1983), cross differentiating (8) and (9) let us obtain the second order differential equation for f in the intermediate region:

$$\mu_b[f_{xx} - k^2 f] + (\mu_b)_x f_x + \frac{k^2 U}{\sigma}(\mu_b)_x f = 0, \quad x \in (-l, 0), \quad (12)$$

where f_x refers to $\mathrm{d}f/\mathrm{d}x$. In the above Sturm–Liouville equation (12), f are the eigenfunctions and $1/\sigma$ are the eigenvalues. We need two boundary conditions to solve the equation (12). We consider that a surface tension S acts on the interface $x = -l$ and a surface tension T acts on $x = 0$. The contact conditions on the interfaces are given by our simplified Laplace's law described in Appendix B. All the details concerning the contact conditions on the interfaces are also given in Gorell and Homsy (1983). We have

$$\mu_b^+(-l) f_x^+(-l) = f(-l)\left(\mu_1 k + \frac{Uk^2}{\sigma}[\mu_1 - \mu_b^+(-l)] + \frac{Sk^4}{\sigma}\right), \quad (13)$$

$$\mu_b^-(0)f_x^-(0) = f(0)\left(-k\mu_2 + \frac{Uk^2}{\sigma}[\mu_2 - \mu_b^-(0)] - \frac{Tk^4}{\sigma}\right), \tag{14}$$

where the superscripts $^-$ and $^+$ stands for the "left" and the "right" limits.

The Sturm–Liouville problem (12)–(14) can be used to study the case of Saffman and Taylor, where no intermediate region is considered, that is $l = 0$. In this case, we have only one interface at $x = 0$ and a surface tension T acts on it. The value of the basic viscosity is μ_1 for $x < 0$ and μ_2 for $x > 0$, therefore, the corresponding eigenfunctions will be exponentials. As perturbations must be zero far enough, Gorell and Homsy (1983) got the well-known formula of Saffman and Taylor (1959):

$$\sigma_{\mathrm{ST}} := \frac{(\mu_2 - \mu_1)Uk - Tk^3}{\mu_2 + \mu_1}. \tag{15}$$

The maximum value σ_{MST} is obtained for the wavenumber k_{MST}:

$$\sigma_{\mathrm{MST}} := \frac{2}{3}\frac{(\mu_2 - \mu_1)U}{\mu_2 + \mu_1}k_{\mathrm{MST}}, \tag{16}$$

$$k_{\mathrm{MST}} := \{(\mu_2 - \mu_1)U/3T\}^{1/2}. \tag{17}$$

As in Gorell and Homsy (1983), we consider the simpler case when the viscosity is continuous on $x = -l$, and no surface tension acts here. We use these conditions for mathematical convenience, but in our case water is immiscible with the polymer. Then our following results can be considered as a first approximation of the case when more general conditions exist on the water – (i.r.) interface.

We have a viscosity jump on the (i.r.) – oil interface only and a surface tension T acts here. Finally, $\mu_b^+(-l) = \mu_1$ and $S = 0$ in (13); then we get a simpler form of this condition:

$$f_x^+(-l) = kf(-l). \tag{18}$$

We use the above relations (16) and (17) to get the following dimensionless quantities, as in Gorell and Homsy (1983):

$$\begin{aligned} &\sigma^* := \tfrac{2\sigma}{3\sqrt{3}\sigma_{\mathrm{MST}}}, \qquad \mu^* := \tfrac{\mu_b}{\mu_1}, \qquad k^* := \tfrac{k}{k_{\mathrm{MST}}\sqrt{3}}, \\ &\alpha := \tfrac{\mu_2}{\mu_1}, \qquad x^* := k_{\mathrm{MST}}x\sqrt{3}, \qquad L := k_{\mathrm{MST}}l\sqrt{3}, \\ &f^* := \tfrac{f}{U}, \qquad \lambda := \tfrac{1}{\sigma^*}, \qquad M := \textstyle\int_{-L}^{0} \mu^*(x^*)\mathrm{d}x^*. \end{aligned} \tag{19}$$

We remark that the maximum value σ^*_{MST} of the dimensionless growth constant σ^*_{ST} is given by

$$\sigma^*_{\mathrm{MST}} := \frac{2}{3\sqrt{3}}. \tag{20}$$

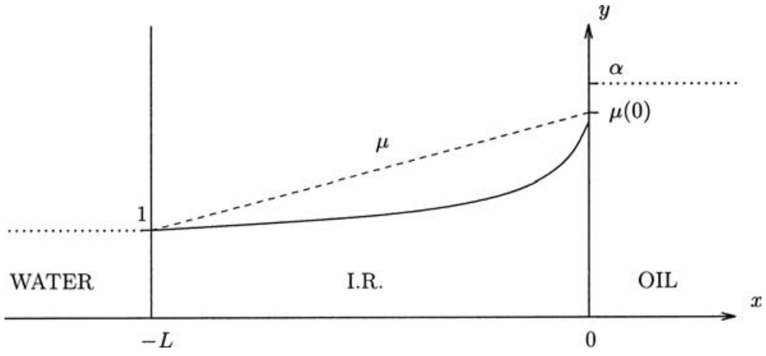

Figure 2. Two examples of viscosity profiles in the I.R.: a linear profile – with dots – and an exponential profile.

In the following, we will omit the superscript $*$ and we will use the notation $df/dx := f'$. Then the relations (12), (14) and (18) give us the problem:

$$\begin{aligned} &-(\mu f')'(x) + k^2\mu(x)f(x) = \lambda k^2 \beta \mu'(x) f(x), \quad x \in (-L, 0),\\ &f'(-L) = (\lambda A + B) f(-L),\\ &f'(0) = (\lambda C + D) f(0), \end{aligned} \tag{21}$$

where

$$\beta := \frac{\alpha + 1}{\alpha - 1}, \tag{22}$$

$$A := 0, \qquad B := k, C := \frac{\beta k^2[\alpha - \mu(0) - k^2(\alpha - 1)]}{\mu(0)}, \tag{23}$$

$$D := -\frac{k\alpha}{\mu(0)}.$$

Our aim is to find viscosities μ which maximize the smallest eigenvalue, i.e. which *minimize* the largest growth constant. In Section 3, we give a theoretical framework of the above problem.

3. Stability Analysis

This section is devoted to a brief study of the above problem (21).

Let a and b be two real numbers such that $a < b$. Let $X := \{f : [a, b] \to \mathbb{C}$ such that $\int_a^b f^2 + (f')^2$ is *finite*$\}$ and $\mu \in C^\infty(a, b)$ be an arbitrary *positive strictly increasing* function on (a, b) accordingly to the property (6). We consider the following Sturm–Liouville eigenvalues problem: find $f \in X$ and $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} & -(\mu f')'(x) + k^2\mu(x)f(x) = \lambda k^2\beta\mu'(x)f(x), \quad x \in (a,b), \\ & f'(a) = (\lambda A + B)f(a), \\ & f'(b) = (\lambda C + D)f(b), \end{aligned} \tag{24}$$

where k and β are real positive numbers. Accordingly to the definition of A, B, C and D in (23) we suppose moreover that

$$A \leqslant 0, \qquad D < 0, \qquad B > 0, \qquad C > 0, \tag{25}$$

and then this problem is more general than (21)-(22)-(23).

3.1. WEAK FORMULATION–RAYLEIGH QUOTIENT

Let define the following bilinear forms $\phi : X \times X \to \mathbb{R}$ and $\psi : X \times X \to \mathbb{R}$ by:

$$\phi(f,g) = \int_a^b \mu f'g' + k^2 \int_a^b \mu fg + B(\mu fg)(a) - D(\mu fg)(b), \tag{26}$$

$$\psi(f,g) = k^2\beta \int_a^b \mu' fg - A(\mu fg)(a) + C(\mu fg)(b), \tag{27}$$

where $(\mu fg)(a)$ stands for $\mu(a)f(a)g(a)$. Then an equivalent form of the problem (24) is (see Courant and Hilbert, 1965):

$$\begin{aligned} & \text{Find } f \in X \text{ and } \lambda \in \mathbb{C} \text{ such that} \\ & \phi(f,g) = \lambda\psi(f,g), \quad \text{for all } g \in X. \end{aligned} \tag{28}$$

Let define $\Phi : X \to \mathbb{R}$ and $\Psi : X \to \mathbb{R}$ by

$$\Phi(f) = \phi(f,f) = \int_a^b \mu(f')^2 + k^2 \int_a^b \mu f^2 + B(\mu f^2)(a) - D(\mu f^2)(b), \tag{29}$$

$$\Psi(f) = \psi(f,f) = k^2\beta \int_a^b \mu' f^2 - A(\mu f^2)(a) + C(\mu f^2)(b). \tag{30}$$

Then Φ and Ψ satisfy the following properties:

(i) Φ is a convex functional (because ϕ is a bilinear positive form),
(ii) Φ is continuous on X,
(iii) Φ is Gâteau differentiable and $d\Phi(f)g = 2\phi(f,g)$,
(iv) Φ is coercive,
(v) Ψ is weakly continuous on X.

Let consider the following minimization problem:

Find $u \in X$ such that $u \neq 0$ and

$$\frac{\Phi(u)}{\Psi(u)} = \inf\left\{\frac{\Phi(f)}{\Psi(f)},\ f \in X,\ f \neq 0\right\}. \tag{31}$$

Properties (i)–(v) imply (see Céa, 1971) that the above problem admits a solution $u \in X$. Following Courant and Hilbert (1965), u is an eigenfunction for the problem (28) corresponding to the smallest eigenvalue $\lambda_R = \Phi(u)/\Psi(u)$, which is a real positive number thanks to hypothesis (25).

3.2. UPPER ESTIMATE OF THE CHARACTERISTIC VALUE

Let λ_R be the smallest eigenvalue of the problem (24) and let define $\sigma_{\mathrm{R}} = 1/\lambda_{\mathrm{R}}$. The following theorem gives an upper bound for σ_{R} in terms of the data of the problem and particularly, in terms of μ.

Theorem 1.

Let $u \in X$ be a solution of (31) and let $\mu \in C^\infty(a,b)$ be a strictly increasing function. Let define

$$\gamma = \sup_{x\in(a,b)} \mu'(x) = \max_{x\in(a,b)} \mu'(x) < +\infty. \tag{32}$$

Then

$$\sigma_{\mathrm{R}} = \frac{\Psi(u)}{\Phi(u)} \leqslant \frac{\max[k^2\beta\gamma, \mu(b)\max(-A, C)]}{\mu(a)\min[k^2, \min(B, -D)]}. \tag{33}$$

Proof. Let u be a solution of (31). As μ is a continuous strictly increasing continuous function

$$\Phi(u) \geqslant \mu(a)(k^2\int_a^b u^2 + Bu^2(a) - Du^2(b)) \tag{34}$$

$$\geqslant \mu(a)\min[k^2, \min(B, -D)](\int_a^b u^2 + u^2(a) + u^2(b)) > 0. \tag{35}$$

Moreover μ' is a continuous function, so

$$\Psi(u) \leqslant k^2\beta\gamma\int_a^b u^2 + \mu(b)(-Au^2(a) + Cu^2(b)) \tag{36}$$

$$\leqslant \max[k^2\beta\gamma, \mu(b)\max(-A, C)](\int_a^b u^2 + u^2(a) + u^2(b)), \tag{37}$$

which ends the proof.

4. Minimizing Viscosity Profiles

We now apply the result of the previous section to the second oil recovery problem. The problem (21) corresponds to the problem (24) with data (22)–(23) and $a = -L$, $b = 0$. The instability on the (i.r.) – oil interface only appears when the limit value of the viscosity on the (i.r.) – oil interface is less than the oil viscosity. Therefore we expect that

$$1 < \mu(0) < \alpha. \tag{38}$$

Moreover, we are only interested in the case when $C > 0$, i.e., when

$$0 < k < \left(\frac{\alpha - \mu(0)}{\alpha - 1}\right)^{1/2} < 1. \tag{39}$$

Indeed, if (39) does not hold, the above problem (24) is not well-defined.

Our aim is to control the maximum characteristic value σ_R. As $C > 0$ and $A := 0$, we get $\max(-A, C) = C$. Condition (38) implies that $\min(B, -D) = k$ and (39) gives us min $[k^2$, min $(B, -D)] = k^2$. Finally, by Theorem (1),

$$\sigma_R \leqslant \frac{\beta}{\mu(-L)} \max[\max_{x \in (-L,0)} \mu'(x), \alpha - \mu(0) - k^2(\alpha - 1)]. \tag{40}$$

A *class of viscosity profiles*. The previous inequality 40 can be used to define a new *class* of viscosity profiles: Indeed, we remark that if μ satisfies the following condition1

$$\max_{x \in (-L,0)} \mu'(x) \leqslant \alpha - \mu(0), \tag{41}$$

then

$$\sigma_R \leqslant \frac{\beta}{\mu(-L)} (\alpha - \mu(0)). \tag{42}$$

We remark that the above upper limit of σ_R is more convenient than the equality (40) because it does not depend on k. Let define the corresponding new maximum growth constant σ_{MR} by

$$\sigma_{MR} = \frac{\beta}{\mu(-L)} (\alpha - \mu(0)). \tag{43}$$

If condition (41) holds, we can choose $\mu(0)$ which gives us a maximum growth constant less than the Saffman–Taylor – dimensionless – value σ^*_{MST} defined by (20). In the following, we describe two families of profiles which satisfy condition (41).

4.1. LINEAR PROFILES

In this section we build a family of linear profiles which belong to the new class of profiles (41) defined above. Let μ_0 and L be real positive numbers satisfying the following conditions:

$$1 < \mu_0 < \alpha, \qquad L \geqslant \frac{\mu_0 - 1}{\alpha - \mu_0}. \tag{44}$$

Then consider the viscosity linear profile, defined on $(-L, 0)$ by

$$\mu(x) = \frac{\mu_0 - 1}{L}(x + L) + 1. \tag{45}$$

Then $\mu(-L) = 1$ and $\mu(0) = \mu_0$. Moreover, following (44), inequality (38) holds and for all $x \in (-L, 0)$ we have

$$\mu'(x) = \frac{\mu_0 - 1}{L} \leqslant \alpha - \mu_0 = \alpha - \mu(0). \tag{46}$$

Finally, condition (41) holds and then, by (42), we get

$$\sigma_R \leqslant \beta(\alpha - \mu_0). \tag{47}$$

Therefore $\mu_0 \to \alpha$ implies $\sigma_R \to 0$. This means that we can choose μ_0 to get a growth constant smaller than the Saffman–Taylor value. The above relation is used to get a lower estimate for the total amount of polymer M:

$$M = \int_{-L}^{0} \mu(x)\,\mathrm{dx} = L\left(\frac{\mu(0) + 1}{2}\right) \geqslant \frac{\mu_0 - 1}{\alpha - \mu_0}\left(\frac{\mu_0 + 1}{2}\right) = \frac{\mu_0^2 - 1}{2(\alpha - \mu_0)}. \tag{48}$$

We can remark that $\mu_0 \to \alpha$ implies $L, M \to \infty$.

4.2. EXPONENTIAL PROFILE

In this section we build a family of "sub-exponential" profiles which satisfy condition (41). Consider a profile μ defined on $(-L, 0)$ such that

$$\mu \text{ is a strictly increasing function,} \tag{49}$$

$$\mu(-L) = 1 \quad \text{and} \quad 1 < \mu(0) < \alpha, \tag{50}$$

$$\frac{\mu'(x)}{\mu(x)} \leqslant \frac{\alpha}{\mu(0)} - 1, \qquad \text{for all } x \in (-L, 0). \tag{51}$$

Following (49), for all $x \in (-L, 0)$, $\mu(x) < \mu(0)$ and then condition (51) implies that, for all $x \in (-L, 0)$, $\mu'(x) \leqslant \alpha - \mu(0)$. This implies that $\max_{x \in (-L,0)} \mu'(x) \leqslant \alpha - \mu(0)$ i.e., condition (41) holds. Finally, we get the same previous result obtained for a linear profile:

$$\sigma_{\mathrm{R}} \leq \beta(\alpha - \mu(0)). \tag{52}$$

Now, we will give some lower estimations for L and M in terms of $\mu(0)$. Let define

$$Q = \frac{\alpha}{\mu(0)} - 1. \tag{53}$$

By integrating both sides of inequality (51), we get

$$\forall x \in (-L, 0), \quad \mu(x) \leq \exp[(x + L)Q]. \tag{54}$$

That is why profiles (49)–(51) are called "sub-exponential". We remark also that $\mu(0) \leq \exp(LQ)$ and then

$$L \geq \frac{1}{Q} \ln \mu(0). \tag{55}$$

Moreover, following conditions (49) and (51) we get trivially

$$M = \int_{-L}^{0} \mu(x)\,\mathrm{d}x > \int_{-L}^{0} \mathrm{d}x = L. \tag{56}$$

Finally,

$$M > L \geqslant \frac{\mu(0) - 1}{\alpha} \ln \mu(0). \tag{57}$$

The previous inequalities implies that if $\mu(0)$ tends to α then L and M tend to infinity.

Example 1. (exponential profile). *Let μ_0 be such that $1 < \mu_0 < \alpha$ and let define*

$$L = \frac{\mu_0}{\alpha - \mu_0} \ln \mu_0. \tag{58}$$

It is easy to prove that the following exponential profile defined on $(-L, 0)$ by

$$\mu(x) = \exp\left[(x + L)\frac{\alpha - \mu_0}{\mu_0}\right], \tag{59}$$

satisfies conditions (49)–(51). *Moreover the corresponding total amount of polymer is given by*

$$M = \frac{\mu_0}{\alpha - \mu_0}(\mu_0 - 1). \tag{60}$$

4.3. COMPARISON WITH PREVIOUS EXPONENTIAL PROFILES

The exponential minimizing viscosity profile in (i.r.) obtained by Carasso and Paşa (1998) depends also on $\mu(0)$, but the condition for the limit value $\mu(0)$ involves L and M. Therefore it was not possible to get some estimations of L and M directly in terms of $\mu(0)$. However, the corresponding maximal growth constant – obtained by Gerschgorin's localization theorem and denoted by σ_G in the following – was obtained directly in terms of $\mu(0)$:

$$\sigma \leqslant \sigma_G = \frac{2\beta}{3\alpha} \frac{(\alpha - \mu(0))^{3/2}}{\sqrt{3(\alpha - 1)}}, \tag{61}$$

for the following exponential profile

$$\frac{\mu'(x)}{\mu(x)} \leqslant \frac{2}{3\alpha} \frac{(\alpha - \mu(0))^{3/2}}{\sqrt{3(\alpha - 1)}}, \qquad x \in (-L, 0). \tag{62}$$

We recall that $\alpha > 1$ and $1 < \mu(0) < \alpha$, then (43) and (61) give us

$$\frac{\sigma_G}{\sigma_{MR}} = \frac{2}{3\sqrt{3}} \frac{\sqrt{\alpha - \mu(0)}}{\alpha\sqrt{\alpha - 1}} < \frac{2}{3\sqrt{3}} \frac{\sqrt{\alpha - \mu(0)}}{\sqrt{\alpha - 1}} < \frac{2}{3\sqrt{3}}, \tag{63}$$

i.e.

$$\sigma_G < \frac{2}{3\sqrt{3}} \quad \sigma_{MR} \text{ where } \sigma_{MR} = \beta(\alpha - \mu(0)). \tag{64}$$

Therefore the upper bound given by Carasso and Paşa (1998) is smaller than the present upper bound: as we consider a more general class of viscosity profiles – we allow variable coefficients in the problem (24) – this result is coherent. However, we emphasize that (52) and (61) give us $\sigma \to 0$ when $\mu(0) \to \alpha$.

4.4. COMPARISON BETWEEN LINEAR AND EXPONENTIAL PROFILE

This section is devoted to the comparison between the (i.r.) lengths and the corresponding amounts of polymer obtained with the linear profile (45) and with the exponential profile (59).

Let L_1 (resp. L_2) be the *smallest suitable* length of the (i.r.) obtained with the linear profile (45) (resp. with the exponential profile (59)), and by M_1 (resp. by M_2) the corresponding total amount of polymer. Then, following (44), (48), (58) and (60),

$$\begin{aligned} L_1 &= \frac{\mu_0 - 1}{\alpha - \mu_0}, \qquad L_2 = \frac{\mu_0}{\alpha - \mu_0} \ln \mu_0, \\ M_1 &= \frac{\mu_0^2 - 1}{2(\alpha - \mu_0)}, \quad M_2 = \frac{\mu_0}{\alpha - \mu_0}(\mu_0 - 1). \end{aligned} \tag{65}$$

Consider $F(x) = 1 + x(\ln x - 1) > 0$, then $F'(x) = \ln(x) > 0, \forall x > 1$. Therefore, F increases for $x > 1$. As $1 < \mu_0 < \alpha$, we obtain

$$L_1 \leqslant L_2. \tag{66}$$

We have also the relation

$$\frac{M_1}{M_2} = \frac{\mu_0 + 1}{2\mu_0} < 1. \tag{67}$$

Consider a *given* μ_0 in $(1, \alpha)$, that is a *given* maximum growth constant σ_{MR}. The two above relations show us that the linear profile (45) gives us smaller lower limits of (i.r.) length and a smaller amount of polymer, compared with the exponential profile (59).

4.5. A GIVEN IMPROVEMENT OF THE STABILITY

The aim of this section is to get a *given* improvement of the stability in the (i.r.) – oil interface – compared with the Saffman Taylor case – by considering above viscosity profiles and by choosing a suitable $\mu(0)$:

Let p be a real number such that $0 < p < 1$ and let define $\mu(0)$ by

$$\mu(0) = \alpha - \frac{2}{\beta 3\sqrt{3}}\, p = \alpha - \frac{2p}{3\sqrt{3}}\frac{\alpha - 1}{\alpha + 1}, \quad 0 < p < 1. \tag{68}$$

As $\alpha > 1 > p > 0$,

$$\frac{2p}{3\sqrt{3}}\frac{1}{\alpha + 1} < 1, \tag{69}$$

and then $\mu(0)$ is such that $1 < \mu(0) < \alpha$. If we consider the linear or exponential profiles defined above, we get

$$\sigma_{\mathrm{MR}} = \beta(\alpha - \mu(0)) = \frac{2p}{3\sqrt{3}} = \sigma_{\mathrm{MST}} \times p, \tag{70}$$

where $0 < p < 1$, that is we get a *given* improvement of a ratio p of the stability.

Once we defined $\mu(0)$ by (68), we get lower estimates for the (i.r.) length L and for the total amount of polymer M, depending on the choice of the viscosity profile:

- in the linear case we get the estimations (44) and (48),
- in the sub-exponential case, we get either the estimation (57) or the relations (58) and (60).

5. Conclusions

The three regions model (1)–(4) was first studied by Gorell and Homsy (1958).

A theoretical formula for an exponential viscosity profile in (i.r.) has been obtained by Carasso and Paşa (1998), according to the numerical results of Daripa *et al.* (1986, 1987, 1988a), Gorell and Homsy (1983), Shan and Sechter (1977) and Uzoigwe *et al.* (1974).

Carasso and Paşa (1998) obtained the maximal growth constant σ_G given by relation (61) by using a finite difference approximation. The value σ_G has been obtained in terms of the limit value of the viscosity on the (i.r.) – oil interface, denoted by $\mu(0)$. The value $\mu(0)$ has been involved in a condition, in terms of the (i.r.) length and the total amount of polymer contained in the (i.r.), denoted respectively by L and M. Therefore it was not possible to estimate L and M directly in terms of $\mu(0)$. Moreover they obtained the following result: if the chosen profile μ satisfies condition (62), then $\mu(0) \to \alpha$ implies that $\sigma_G \to 0$, where α is the ratio between the oil and the water viscosities.

In the present paper, we suggest a new estimate (42) of the *exact* growth constant σ_R. This estimate is also in terms of the above limit value $\mu(0)$ which is an arbitrary value in $[1, \alpha]$. To prove this result, we use the Rayleigh quotient (31) of the initial Sturm–Liouville problem (12)–(14)–(18) *without any numerical treatment*.

We use relation (40) to get *a class* of theoretical profiles of viscosity in (i.r.), characterized by condition (41). This condition allows us to obtain the general formula (43) for the maximum growth constant. Condition (41) gives us two particular minimizing viscosity profiles: the *linear* profiles defined by (44)–(45) and the *sub-exponential* profiles satisfying (49)–(51).

The linear profiles allows us to consider variable coefficients in the problem (12)–(14)–(18), while the exponential case (59) gives us only constant coefficients, as in Carasso and Paşa (1998).

For both linear and exponential profiles we obtain the same maximal value σ_{MR} of the growth constant, given by (43). We have $\sigma_{\mathrm{MR}} \to 0$ when $\mu(0) \to \alpha$, as in Carasso and Paşa (1998). Therefore the formula (43) and a value $\mu(0)$ close enough to α give us an improved stability, compared with the Saffman–Taylor case.

In Sections 4.1 and 4.2 we obtain lower estimations of the (i.r.) length L and of the amount of polymer M, directly in terms of $\mu(0)$, then we generalize the previous result of Carasso and Paşa. We have the relations (44) and (48) for the linear case and the relations (55) and (56) for the sub-exponential case.

In Section 4.3, we give the comparison (64) between the present maximal growth constant σ_{MR} (52) and the previous maximal value σ_{G} (61). We have $\sigma_{\mathrm{G}} \leqslant \sigma_{\mathrm{MR}}$. This is not surprising, because we consider now a more general case, including variable coefficients in the stability problem (12)–(14)–(18).

In Section 4.4 we prove that the linear case is more favorable: Indeed, we need smaller L and M to get the same growth constant σ_{R} compared to the exponential case.

In Section 4.5 we compute L and M corresponding to *a given improvement of the stability*, compared with the Saffman–Taylor case. For this we define $\mu(0)$ by (68) and then σ_{MR} verifies (70), where σ_{MST} is the dimensionless maximal growth constant of Saffman and Taylor. Once $\mu(0)$ is fixed, some estimations of L and M are given, in terms of α and p – the *given* ratio of the improvement – depending on the considered profile:

- in the linear case we get the estimations (44) and (48),
- in the sub-exponential case, we get either the estimation (57) or the relations (58) and (60).

In conclusion, this paper suggests and describes a new class of viscosity profiles which minimize the maximal growth constant of the perturbations in the three regions model considered by Gorell and Homsy (1983). This class of profiles was obtained by considering an upper bound for the maximal characteristic value of a Sturm–Liouville problem by using the Rayleigh quotient *without any discretization*. This class, which contains linear and sub-exponential profiles, is coherent with previous theoretical and numerical results. We emphasize that the linear case is more favorable than the exponential one regarding the corresponding needed amount of polymer. Finally we compute the (i.r.) length and the amount of polymer corresponding to a given improvement of stability, compared with the Saffman–Taylor case – i.e., without (i.r.).

Appendix A. The Hele–Shaw Model

In Lamb (1932) is given a short description of the Hele–Shaw model. A flow of a liquid between two parallel plates separated from a distance h is considered. The flow is two-dimensional in the plane xOy and the Oz direction is perpendicular to the plane. The velocity is denoted by (u, v, w) and p is the pressure. A Stokes fluid is considered and $w = 0$, therefore p is not depending on z. The derivatives of (u, v) with respect to x, y are neglected, compared with the z derivative. Then, we have

$$\mu \frac{\partial^2 u}{\partial^2 z} = \frac{\partial p}{\partial x}, \qquad \mu \frac{\partial^2 v}{\partial^2 z} = \frac{\partial p}{\partial y},$$

where μ is the viscosity. The non-slipping condition is imposed for $z = 0, z = h$ and we get

$$\overline{u} = -\frac{h^2}{12}\frac{1}{\mu}\frac{\partial p}{\partial x}, \qquad \overline{v} = -\frac{h^2}{12}\frac{1}{\mu}\frac{\partial p}{\partial y}, \tag{A.1}$$

where

$$\overline{u} = \frac{1}{h}\int_0^h u(x, y, z)\,\mathrm{d}z, \qquad \overline{v} = \frac{1}{h}\int_0^h v(x, y, z)\,\mathrm{d}z.$$

Equation (A.1) can be obtained by using second order Taylor expansions for u, v and by the fact that the derivatives of p are not depending on z.

Equation (A.1) is similar with Darcy's law of a flow in a two-dimensional porous medium with permeability $h^2/12$, where the filtration velocity is given by $\overline{u}$ and $\overline{v}$. We emphasize that only the permeability and the velocity of fluid are "fictive", but the viscosity in Equation (A.1) is the same as in Darcy's law.

Appendix B. Laplace's Law and Hele–Shaw Model

We consider two immiscible viscous fluids between two parallel plates, separated from a distance h as in the above model. We can consider that the porous equivalent medium is divided in two regions: region I, filled by the fluid 1 with a viscosity μ_1 and region II filled by the fluid 2 with the viscosity μ_2. These regions are separated by a sharp interface Γ, which form a meniscus of angle θ with the plates. The pressures of the fluids are denoted by p_1 and p_2, respectively. Laplace's law can be considered of the following form:

$$p^+ - p^- = T\left(2\frac{\cos(\theta)}{h} + \frac{1}{R}\right),$$

where the superscripts $-$ and $+$ stand, respectively, for the left and the right limits in the neighborhood of Γ. R is the curvature radius of Γ in the plane xOy. We neglect the term containing the contact angle θ. More details are given in Paşa (2002).

We emphasize that in this flow model, every point of region I only contains the fluid 1; every point of the region II only contains the fluid 2. Therefore, the saturation of the fluid 1 is 1 in the region I and 0 in the region II; the saturation of the fluid 2 is 0 in the region I and 1 in the region II. For incompressible fluids, we must consider also the continuity equation for the velocities (u, v). Moreover, the relative permeabilities of the two fluids are constant in the two regions.

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References

Carasso, C. and Paşa, G.: 1998, An optimal viscosity profile in the secondary oil recovery, *Math. Modeling Num. Anal. (M2AN)* **32**(2), 211–221.

Carasso, C. and Paşa, G.: 2000, A modified green function for a control problem in oil recovery, *Comput. Meth. Appl. Mech. Eng.* **190**(8–10), 1197–1207.

Céa, J.: 1971, *Optimisation – Théorie et algorithmes*. Dunod, Paris.

Chouke, R. L., Van Der Meurs, P. and Van Der Poel, C.: 1959, The stability of a slow, immiscible, viscous liquid–liquid displacement in a permeable media, *Trans AIME* **216**, 188–194.

Courant, R. and Hilbert, D.: 1965, *Methods of Mathematical Physics*, Intersciences Publishers Inc., New York.

Daripa, P., Glimm J., Grove., J., Linnquist, B. and McBryan, O.: 1986, Reservoir simulation by the method of front tracking, in: *Proceedings of the IFE/SSI seminar on Reservoir Description and Simulation with Emphasis on EOR*, Oslo, 18 p.

Daripa, P.: 1987, Instability and its control in oil recovery problem, in: *Proceedings of 6th IMACS International Symposium on Computer Methods for Part. Diff. Eq. – VI,* ed. R. Vichnevetsky, Bethellem PA., pp. 411–418.

Daripa, P., Glimm, J., Lindquist, B. and McBryan, O.: 1988a, Polymer floods: A case study of nonlinear wave analysis and instability control in tertiary oil recovery, *SIAM J. Appl. Math.* **49**, 353–373.

Daripa, P., Glimm, J., Lindquist, B., Maesumi, M. and McBryan, O.: 1988b, On the simulation of heterogeneous petroleum reservoirs, *Numerical Simulation in Oil Recovery, IMA Vol. Math. Appl. 11*, ed. M. Wheeler et al., Springer, New York, 89–103.

Gorell, S. B. and Homsy, G. M.: 1983, A theory of the optimal policy of oil recovery by secondary displacement process, *SIAM J. Appl. Math.* **43**(1), 79–98.

Lamb, S. H.: 1932, *Hydrodynamics*, *Cambridge University Press*, Section 330.

Marle, C. M.: 1981, *Multiphase Flow in Porous Media*, Culf Publishing Company, Houston.

Mungan, M.: 1971, Improved waterflooding through mobility control, *Canad. J. Chem. Engr.* **49**, 32–37.

Pearson, H. J.: 1977, The stability of some variable viscosity flows with applications to oil extraction, *Cambridge Univ. Report*.

Paşa, G.: 1996, An existence theorem for a control problem in oil recovery, *Num. Funct. Anal. Optimiz.* **17**(9–10), 911–923.

Paşa, G. and Poliševski D.: 1992, Instability of interfaces in oil recovery, *Int. J. Eng. Sci.* **30**(2), 161–167.

Paşa, G.: 2002, A new optimal growth constant for the Hele-Shaw instability, *Trans. Porous Med.* **49**(1), 27–40.

Saffman, P. G., and Taylor, G. I.: 1959, The penetration of a fluid in a porous medium or Helle-Shaw cell containing a more viscous fluid, *Proc. Roy. Soc.* A **245**, 312–329.

Shah, D. and Schecter, R.: 1977, *Improved Oil Recovery by Surfactants and Polymer Flooding*, Academic Press, New York.

Uzoigwe, A. C., Scanlon, F. C. and Jewett, R. L.: 1974, Improvement in polymer flooding: The programmed slug and the polymer-conserving agent, *J. Petrol. Tech.* **26**, 33–41.

Zhan, L. and Yortsos, Y. C.: 2002, The shape of a gravity finger in a rectangular channel in homogeneous porous media, *Trans. Porous Med.* **49**, 77–97.