Reduction of computation in the numerical resolution of a second kind weakly singular Fredholm equation

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1. Introduction

X denotes the Banach space $L^1([0,\tau^\star])$ of all — equivalence classes of — Lebesgue integrable functions on $[0,\tau^\star]$, where $\tau^\star$ is a given non negative large number. We consider an integral operator $T : X \to X$ defined by

$$x \mapsto Tx \ : \ \tau \in [0,\tau^\star] \mapsto (Tx)(\tau) = \int_0^{\tau^\star} \kappa(\tau,\sigma)x(\sigma)d\sigma.$$

We suppose that the kernel κ is of the following form:

$$\kappa(\tau,\sigma) := \eta(\tau,\sigma)g(|\tau-\sigma|),$$

where η is a continuous complex valued function on the square $[0,\tau^\star] \times [0,\tau^\star]$ and g is a weakly singular function at zero, in the following sense:

(a) $\lim_{\tau\to 0^+} g(\tau) = +\infty$;

(b) $g \in C^0(]0,\tau^\star]) \cap L^1([0,\tau^\star])$;

(c) g is a positive decreasing function on $]0,\tau^\star]$.

It was proved in [1] that T is a compact operator in X. For $z \in \mathrm{re}(T) := \{z \in \mathbb{C} \ : \ T - zI \text{ is bijective}\}$ and $f \in X$, there is a unique solution φ to the problem

$$(T - zI)\varphi = f.$$

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2. Finite rank approximations

We study a special class of approximate operators whose range is a finite n-dimensional subspace of X. For all $\ell \in X^*$, the topological adjoint space of X, and all $x \in X$, we define $\langle x, \ell \rangle := \overline{\ell}(x)$. As ℓ is linear conjugate, $\langle x, \ell \rangle$ is linear with respect to x and linear conjugate with respect to ℓ, such as a scalar product of a complex prehilbertian space. We recall that a linear bounded finite rank operator is compact and can be written as (see [2]):

$$T_n := \sum_{j=1}^{n} \langle \, . \, , \ell_{n,j} \rangle e_{n,j},$$

where $n \in \mathbb{N}^*$ and, for $j \in [\![1, n]\!]$, $\ell_{n,j} \in X^*$ and $e_{n,j} \in X$. If $z \in \mathrm{re}(T_n)$, then for all $f \in X$, the approximate equation

$$(T_n - zI)\varphi_n = f \tag{1}$$

admits a unique solution. If we set

$$\mathrm{A}_n(i, j) := \langle e_{n,j}, \ell_{n,i} \rangle, \; \mathrm{b}_n(i) := \langle f, \ell_{n,i} \rangle \; \text{ and } \mathrm{x}_n(j) := \langle \varphi_n, \ell_{n,j} \rangle, \tag{2}$$

equation (1) leads to the following n-dimensional linear system, by applying each semi-linear functional $\ell_{n,i}$ to each member of equation (1):

$$(\mathrm{A}_n - z\mathrm{I}_n)\mathrm{x}_n = \mathrm{b}_n, \tag{3}$$

where I_n denotes the identity matrix of order n.

We are interested in approximations obtained with a family of projections on X with finite rank n. Such a projection π_n reads as

$$\pi_n x := \sum_{j=1}^{n} \langle x, \xi_{n,j} \rangle e_{n,j}, \quad x \in X,$$

where $(e_{n,j})_{j=1}^n$ is an ordered basis of the range of π_n and $(\xi_{n,j})_{j=1}^n$ is an adjoint basis of the former. The projection π_n considered in this paper is built as follows: let $(\tau_{n,i})_{i=1}^n$ be a grid on $[0, \tau^\star]$ such that

$$0 := \tau_{n,0} < \tau_{n,1} < \ldots < \tau_{n,n-1} < \tau_{n,n} := \tau^\star.$$

Set $h_{n,j} := \tau_{n,j} - \tau_{n,j-1}$ for $j \in [\![1, n]\!]$ and define

$$\mu_n := \min\{h_{n,j}, \quad j \in [\![1, n]\!]\}, \qquad h_n := \max\{h_{n,j}, \quad j \in [\![1, n]\!]\}.$$

We define for all $x \in X$ and for all $j \in [\![1, n]\!]$

$$\langle x, \xi_{n,j} \rangle := \frac{1}{h_{n,j}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} x(\sigma) d\sigma,$$

and for all $\tau \in [0, \tau^\star]$

$$e_{n,j}(\tau) := \begin{cases} 1 & \text{if } \tau \in]\tau_{n,j-1}, \tau_{n,j}[, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $x \in C^0([0, \tau^\star])$ then, for all $j \in [\![1, n]\!]$, $\lim_{h_n \to 0} \langle x, \xi_{n,j} \rangle = x(\tau_{n,j})$. We define the approximate operator T_n by

$$T_n := \pi_n T.$$

It was proved in [1] that $(\pi_n)_{n \geq 1}$ is pointwise convergent to the identity operator in X. Then as T is compact, the sequence $(T_n)_{n \geq 1}$ is convergent to T in the operator norm. The entries of the matrix A_n and of the second member b_n defined by (2) are given for all $(i, j) \in [\![1, n]\!]^2$ by

$$\begin{aligned} \mathrm{A}_n(i,j) &= \frac{1}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} (Te_{n,j})(\tau) d\tau = \frac{1}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} \kappa(\tau, \sigma) d\sigma d\tau, \\ \mathrm{b}_n(i) &= \frac{1}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} \int_0^{\tau^\star} \kappa(\tau, \sigma) f(\sigma) d\sigma d\tau. \end{aligned} \tag{4}$$

3. Reduction of computation

To attain a given precision on the approximate solution φ_n, it may be necessary that the largest grid step h_n be so small that the dimension of the corresponding linear system will be prohibitively large from a computational point of view so as the matrix A_n is full. Moreover, when g is — for instance — an exponentially decreasing function, a lot of entries of A_n are very close to zero. Here, we suggest a consistent way to reset to zero some small entries: in fact, we prove that a truncation on the kernel of T induces the zeroing of some small entries in absolute value of the matrix A_n.

Consider a sequence $(\varepsilon_n)_{n \geq 0}$ of positive real numbers. We define

$$I_{n,i,j} := [\tau_{n,i-1}, \tau_{n,i}] \times [\tau_{n,j-1}, \tau_{n,j}], \qquad (i,j) \in [\![1, n]\!]^2,$$

and

$$\mathcal{E}_n := \left\{ (i,j) \in [\![1, n]\!]^2, \sup_{(t,s) \in I_{n,i,j}} |\kappa(t,s)| \leq \frac{\varepsilon_n}{h_{n,j}} \right\}.$$

Let κ_n be the function defined for all $(\tau, \sigma) \in [0, \tau^\star] \times [0, \tau^\star]$ such that $\tau \neq \sigma$, by

$$\kappa_n(\tau, \sigma) := \begin{cases} 0 & \text{if } \exists (i,j) \in \mathcal{E}_n \text{ such that } (\tau, \sigma) \in I_{n,i,j}, \\ \kappa(\tau, \sigma) & \text{otherwise.} \end{cases}$$

Let K_n be the integral operator induced by the kernel κ_n,

$$x \mapsto K_n x \;:\; \tau \in [0, \tau^\star] \mapsto (K_n x)(\tau) = \int_0^{\tau^\star} \kappa_n(\tau, \sigma) x(\sigma) d\sigma,$$

and consider the finite rank approximation

$$\widetilde{T}_n := \pi_n K_n.$$

Let us denote by $\widetilde{\mathrm{A}}_n$ the matrix of the linear system (3) corresponding to the approximation $\widetilde{T}_n$, i.e.

$$\widetilde{\mathrm{A}}_n(i,j) = \frac{1}{h_{n,i}} \iint_{I_{n,i,j}} \kappa_n(\tau,\sigma) d\tau d\sigma, \quad (i,j) \in [\![1,n]\!]^2. \tag{5}$$

The following theorem shows that the truncation of κ induces the zeroing of some entries of A_n which are less than ε_n in absolute value.

Theorem 1. *Let* A_n *and* $\widetilde{\mathrm{A}}_n$ *be the matrices defined by (4) and (5) respectively. Then for all* $(i,j) \in \mathcal{E}_n$, $|\mathrm{A}_n(i,j)| \leq \varepsilon_n$, *and for all* $(i,j) \in [\![1,n]\!]^2$,

$$\widetilde{\mathrm{A}}_n(i,j) = \begin{cases} 0 & \text{if } (i,j) \in \mathcal{E}_n, \\ \mathrm{A}_n(i,j) & \text{otherwise.} \end{cases}$$

Proof. Evident.

We now justify the use of $\widetilde{T}_n$ instead of T_n in the approximate equation (1).

Theorem 2. *If* $(\varepsilon_n)_{n\geq 0}$ *is such that* $\lim_{n\to+\infty} \frac{\varepsilon_n}{\mu_n} = 0$ *then* $\widetilde{T}_n$ *converges to* T *in the operator norm.*

Proof. Let $x \in L^1([0,\tau^\star])$. Then,

$$\begin{aligned} \|(T-K_n)x\|_1 &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} \iint_{I_{n,i,j}} |\kappa(\tau,\sigma) - \kappa_n(\tau,\sigma)||x(\sigma)| d\sigma d\tau \\ &\leq \sum_{(i,j)\in\mathcal{E}_n} \frac{\varepsilon_n}{h_{n,j}} \iint_{I_{n,i,j}} |x(\sigma)| d\sigma d\tau \leq \sum_{(i,j)\in\mathcal{E}_n} \frac{\varepsilon_n h_{n,i}}{h_{n,j}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} |x(\sigma)| d\sigma \leq \frac{\tau^\star \varepsilon_n}{\mu_n} \|x\|_1. \end{aligned} \tag{6}$$

Since $\lim_{n\to+\infty} \frac{\varepsilon_n}{\mu_n} = 0$, $\lim_{n\to+\infty} \|T - K_n\|_1 = 0$. Also,

$$\|T - \widetilde{T}_n\|_1 \leq \|T-K_n\|_1 + \|(I-\pi_n)T\|_1 + \|(I-\pi_n)(K_n - T)\|_1,$$

and the conclusion follows.

Remark 1. Note that it is not necessary to compute the entries before their zeroing: the condition which defines $\mathcal{E}_n$ can be used to decide the zeroing of an entry. In the case of a large matrix, this trick allows a gain of time in the construction of the matrix. Finally, note that some entries whose absolute value is less than ε_n may not be zeroed.

4. Numerical example

In this section, we apply the previous reduction of computation to the resolution of an integral equation which appears in a radiative transfer problem. A description of the physical problem is given in [3] and [4]. A parallel code for the resolution of this problem is given in [5]. Let κ and f be defined for all $(\tau, \sigma) \in [0, \tau^\star]^2$ by

$$\kappa(\tau,\sigma) := \frac{\varpi_\star}{2} E_1(|\tau-\sigma|) = \frac{\varpi_\star}{2}\int_0^1 \frac{\exp(-|\tau-\sigma|/\mu)}{\mu} d\mu, \quad \tau \neq \sigma,$$
$$f(\tau) := \begin{cases} \varpi_\star - 1 & \text{if } \tau \in [0, \tau^\star/2], \\ 0 & \text{otherwise,} \end{cases}$$

where $\varpi_\star \in]0,1[$. This kernel satisfies conditions (a), (b) and (c). The corresponding entries of A_n and b_n are given in [5].

We set $\varpi_\star = 0.75$ and we used an uniform grid of 100 nodes on $[0, \tau^\star]$ to compute the results shown in the following figures.

In Fig. 1 we show the profile of the corresponding matrix $\widetilde{\mathrm{A}}_n$ when $\epsilon_n = 10^{-12}$: in this case, 5853 entries have been zeroed in the matrix A_n, i.e. more than 50%.

Let $\widetilde{\varphi}_n$ be the solution of

$$(\widetilde{T}_n - zI)\widetilde{\varphi}_n = f. \tag{7}$$

Then in Fig. 2 we show the — log-scaled — relative error $\dfrac{\|\varphi_n - \widetilde{\varphi}_n\|_1}{\|f\|_1}$ for different values of ε_n. Note that the variation of the relative error is linear and that the numerical results are consistent with respect to the bound (6).

Remark 2. Keep in mind that the entries of the second member of the linear system (3) corresponding to (7) are given for all $i \in [\![1, n]\!]$ by:

$$\widetilde{\mathrm{b}}_n(i) := \frac{1}{h_{n,i}} \sum_{\substack{j \in [\![1,n]\!] \\ (i,j) \notin \mathcal{E}_n}} \iint_{I_{n,i,j}} \kappa(\tau,\sigma) f(\sigma) d\sigma.$$

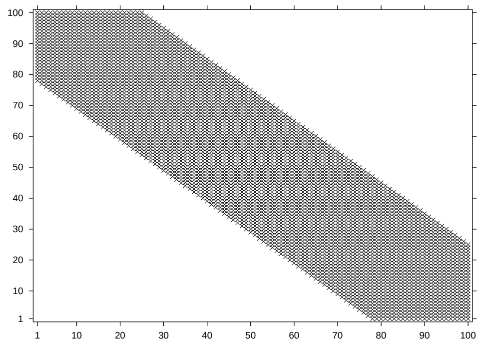

Fig. 1. Profile of the truncated matrix $\widetilde{\mathrm{A}}_n$ when $\varepsilon_n = 10^{-12}$.

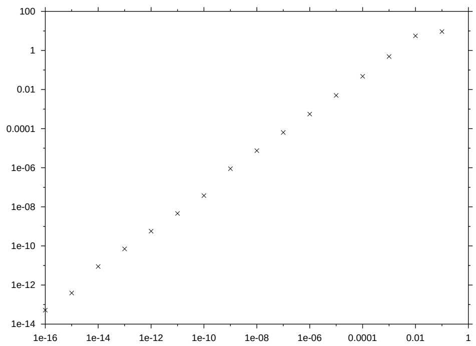

Fig. 2. $\varepsilon_n \mapsto \dfrac{\|\varphi_n - \widetilde{\varphi}_n\|_1}{\|f\|_1}$. (Log scale.)

References

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