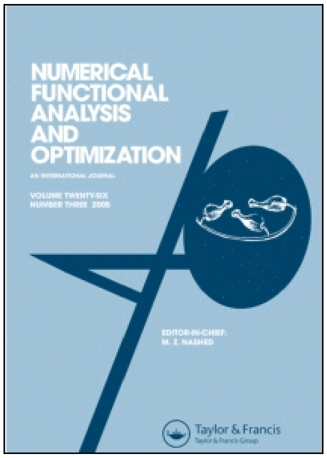

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THE ROLES OF A WEAK SINGULARITY AND THE GRID UNIFORMITY IN RELATIVE ERROR BOUNDS

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ABSTRACT

We develop a relative error bound specifying the role of the singularity and the one of the grid uniformity in the case of projection type approximations for the solution of a Fredholm integral equation of the second kind with a weakly singular kernel of convolution type. Numerical experiments with equations having solutions of different nature, namely continuous, integrable bounded discontinuous and integrable unbounded, complete this work. These experiments give an idea on how realistic the theoretical relative error estimates are, depending on the nature of the solution.

1. INTRODUCTION

Both $X := C^0([0,1])$ or $X := L^1([0,1])$ are complex Banach spaces X which can be used as theoretical framework for an integral operator $T: X \to X$ defined by

$$(Tx)(\tau) := \int_0^1 g(|\tau - \tau'|)x(\tau')\, d\tau' \tag{1}$$

where the kernel g is weakly singular in the following sense:

$$\lim_{\tau \to 0^+} g(\tau) = +\infty, \tag{2}$$

$$g \in C^0(]0,1]) \cap L^1([0,1]), \tag{3}$$

$$g(\tau) \geq 0 \text{ for all } \tau \in]0,1], \tag{4}$$

$$g \text{ is a decreasing function on }]0,1]. \tag{5}$$

An $L^2([0,1])$ analysis is developed in [5]. Other references on this subject are [2], [3], [7], [8], and [10].

The following function $G\colon [0,1] \to \mathbb{R}$ will play an important technical role throughout the paper:

$$G(\tau) := \int_0^\tau g(\tau')\, d\tau'. \tag{6}$$

Theorem 1. *The operator* T *is compact both in the space* $C^0([0,1])$ *and in the space* $L^1([0,1])$*, and*

$$\|T\|_1 = \|T\|_\infty = 2\int_0^{1/2} g(\tau)\, d\tau. \tag{7}$$

Proof. We leave to the reader the proof of that fact that both $C^0([0,1])$ and $L^1([0,1])$ are invariant under T.

Let $y:\ [0,1] \to \mathbb{R}$ be defined by

$$y(\tau) := \int_0^1 g(|\tau' - \tau|)\, d\tau' = G(\tau) + G(1-\tau).$$

The function y is symmetric with respect to $1/2$, and

$$y'(\tau) = g(\tau) - g(1-\tau) \begin{cases} > 0 & \text{if } 0 < \tau < 1/2, \\ < 0 & \text{if } 1/2 < \tau < 1, \end{cases}$$

hence

$$\|T\|_\infty = \max_{0 \leq \tau \leq 1} y(\tau) = y(1/2) = 2\int_0^{1/2} g(\tau)\, d\tau.$$

This proves that T is bounded from $C^0([0,1])$ into itself.

Also, for all $x \in L^1([0,1])$ such that $\|x\|_1 \leq 1$,

$$\|Tx\|_1 = \int_0^1 \left| \int_0^1 g(|\tau' - \tau|) x(\tau)\, d\tau \right| d\tau' \leq \sup_{\tau \in [0,1]} \int_0^1 g(|\tau - \tau'|)\, d\tau',$$

and hence $\|T\|_1 \leq \|T\|_\infty$, so T is bounded from $L^1([0,1])$ into itself. Define, for each integer $n \geq 2$, the function $x_n \in L^1([0,1])$ by

$$x_n(\tau) := \begin{cases} 0 & \text{if } |\tau - 1/2| > 1/n, \\ \dfrac{n}{2} & \text{otherwise.} \end{cases}$$

Then, for each integer $n \geq 2$, there exists $\widetilde{\tau}_n$ such that $|1/2 - \widetilde{\tau}_n| < 1/n$ and

$$\|Tx_n\|_1 = \frac{n}{2}\int_0^1 \int_{1/2-1/n}^{1/2+1/n} g(|\tau' - \tau|)\, d\tau\, d\tau' = \int_0^1 g(|\tau' - \widetilde{\tau}_n|)\, d\tau',$$

so

$$\lim_{n\to\infty} \|Tx_n\|_1 = \int_0^1 g(|\tau' - 1/2|)\, d\tau' = 2\int_0^{1/2} g(\tau)\, d\tau = \|T\|_\infty = \|T\|_1.$$

Compactness follows by considering the truncated kernel g_n defined by

$$g_n(\tau) := \begin{cases} g(1/n) & \text{if } \tau < 1/n, \\ g(\tau) & \text{otherwise.} \end{cases}$$

The corresponding integral operator T_n defined by

$$(T_n x)(\tau) := \int_0^1 g_n(|\tau - \tau'|) x(\tau')\, d\tau'$$

is compact either from $C^0([0,1])$ into itself or from $L^1([0,1])$ into itself since g_n is a continuous function.

For $x \in C^0([0,1])$ or $x \in L^1([0,1])$ we have

$$\begin{aligned}(T - T_n)x(\tau) &= \int_{\tau-1/n}^{\tau} [g(\tau - \tau') - g(1/n)]x(\tau')\, d\tau' \\ &\quad + \int_{\tau}^{\tau+1/n} [g(\tau' - \tau) - g(1/n)]x(\tau')\, d\tau' \\ &= \int_0^{1/n} [g(t) - g(1/n)]x(\tau - t)\, dt \\ &\quad + \int_0^{1/n} [g(t) - g(1/n)]x(\tau + t)\, dt.\end{aligned}$$

Hence, in $C^0([0,1])$,

$$\|T - T_n\|_\infty \leq 4\int_0^{1/n} g(\tau)\, d\tau \;\to\; 0 \quad \text{as } n \to \infty,$$

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and in $L^1([0,1])$, x being extended by 0 outiside $[0,1]$,

$$\begin{aligned}\int_0^1 \left|(T - T_n)x(\tau)\right| d\tau &\leq \int_0^1 \int_0^{1/n} |g(t) - g(1/n)|[|x(\tau - t)| + |x(\tau + t)|]\, dt\, d\tau \\ &= \int_0^{1/n} |g(t) - g(1/n)| \int_0^1 [|x(\tau - t)| + |x(\tau + t)|]\, d\tau\, dt \\ &= \int_0^{1/n} |g(t) - g(1/n)| \left(\int_{-t}^{1-t} |x(s)|\, ds + \int_t^{1+t} |x(s)|\, ds \right) dt \\ &\leq 4\|x\|_1 \int_0^{1/n} g(t)\, dt,\end{aligned}$$

so

$$\|T - T_n\|_1 \leq 4 \int_0^{1/n} g(\tau)\, d\tau \quad \to \quad 0 \quad \text{as } n \to \infty.$$

Hence, in both spaces, T is the uniform limit of a sequence of compact operators. ■

2. PROJECTION APPROXIMATIONS

In this section, X denotes a general complex Banach space not reduced to $\{0\}$, and T is a linear compact operator in X. Both the norm in X and the subordinated norm for linear bounded operators in X will be denoted by $\|\cdot\|$.

For z in the resolvent set of T, we consider projection approximations for the Fredholm equation

$$T\varphi = z\varphi + f. \tag{8}$$

Since T is compact and the space X is not reduced to $\{0\}$, we have $z \neq 0$ and the equation (8) is of the second kind.

We recall that a bounded linear finite rank operator T_n in X can be written as

$$T_n := \sum_{j=1}^{n} \langle \cdot, \ell_{n,j} \rangle e_{n,j} \tag{9}$$

where $n \geq 1$ is an integer, and, for $j \in [\![1, n]\!]$, $\ell_{n,i} \in X^*$, the adjoint space of X, and $e_{n,j} \in X$.

The resolution of the approximate equation

$$T_n\varphi_n = z\varphi_n + f, \tag{10}$$

where z belongs to the resolvent set of T_n, leads to an n-dimensional linear system since equation (10) reads as

$$\sum_{j=1}^{n} \langle \varphi_n, \ell_{n,j} \rangle e_{n,j} - z\varphi_n = f \tag{11}$$

and applying $\ell_{n,i}$ we get

$$\sum_{j=1}^{n} \langle \varphi_n, \ell_{n,j} \rangle \langle e_{n,j}, \ell_{n,i} \rangle - z\langle \varphi_n, \ell_{n,i} \rangle = \langle f, \ell_{n,i} \rangle \tag{12}$$

that is, the system in the unknown vector x,

$$(\mathsf{A} - z\mathsf{I})\mathsf{x} = \mathsf{b} \tag{13}$$

where

$$\mathsf{A}(i,j) := \langle e_{n,j}, \ell_{n,i} \rangle, \tag{14}$$

$$\mathsf{b}(i) := \langle f, \ell_{n,i} \rangle, \tag{15}$$

$$\mathsf{x}(j) := \langle \varphi_n, \ell_{n,j} \rangle. \tag{16}$$

Once this system is solved, the solution of (10) is given by

$$\varphi_n = \frac{1}{z}\left(\sum_{j=1}^{n} \mathsf{x}(j) e_{n,j} - f \right), \tag{17}$$

which is another way of writing (10).

We are interested in approximations of the form

$$T_n := \pi_n T, \tag{18}$$

where $(\pi_n)_{n\geq 1}$ is a sequence of projections with finite rank n.

A projection π_n of finite rank n is defined by

$$\pi_n x := \sum_{j=1}^{n} \langle x, e^*_{n,j} \rangle e_{n,j}, \quad x \in X, \tag{19}$$

where $(e_{n,j})_{j=1}^{n}$ is an ordered basis of the range of π_n, and $(e^*_{n,j})_{j=1}^{n}$ is an adjoint basis of the former. The intersection of the kernels of these semi-linear functionals gives the kernel of π_n. Thus, the approximation of T induced by the projection π_n satisfies, for all $x \in X$,

$$T_n x = \pi_n T x = \sum_{j=1}^{n} \langle Tx, e^*_{n,j} \rangle e_{n,j}. \tag{20}$$

Identifying $\ell_{n,j} := T^* e^*_{n,j}$, we find the representation (9).

We suppose that $(\pi_n)_{n\geq 1}$ is pointwise convergent to the identity operator in the Banach X where the operator T is defined and since T is compact, this implies the convergence of $(T_n)_{n\geq 1}$ to T in the norm of the Banach algebra of all bounded linear operators in X. Let us set

$$R(z) := (T - zI)^{-1}. \tag{21}$$

Since

$$\lim_{n\to\infty} \|T_n - T\| = 0, \tag{22}$$

then

$$R_n(z) := (T_n - zI)^{-1} \tag{23}$$

exists for n large enough and is uniformly bounded. That is, there exists n_0 such that

$$c_0(z) := \max\{\|R(z)\|, \sup_{n>n_0} \|R_n(z)\|\} < +\infty. \tag{24}$$

Hence

$$\|\varphi - \varphi_n\| = \|R_n(z)(T_n - T)R(z)f\| \le c_0(z)\|(T_n - T)\varphi\| \tag{25}$$

$$= c_0(z)\|(I - \pi_n)T\varphi\|. \tag{26}$$

Let us denote

$$X_n := \pi_n X \tag{27}$$

the range of π_n. For all $x \in X$ and all $y \in X_n$ we have

$$\|(I - \pi_n)x\| = \|x - y + \pi_n(y - x)\| \le (1 + \|\pi_n\|)\|x - y\| \tag{28}$$

and hence

$$\|(I - \pi_n)x\| \le (1 + \|\pi_n\|)d(x, X_n), \tag{29}$$

where

$$d(x, X_n) := \inf_{y\in X_n} \|x - y\|. \tag{30}$$

Thus

$$\|\varphi - \varphi_n\| \le c_0(z)(1 + \|\pi_n\|)d(T\varphi, X_n). \tag{31}$$

This estimate will be developed in the space $C^0([0,1])$ in Section 3.

Also,

$$\begin{aligned}\|\varphi - \varphi_n\| &= \|R(z)(T_n - T)R_n(z)f\| \\ &\le \|R(z)\|\|(T_n - T)\varphi_n\| \\ &= \|R(z)\|\|(I - \pi_n)T\varphi_n\| \\ &\le \|R(z)\|\|(I - \pi_n)T\pi_n\varphi_n\| \\ &\quad + \|R(z)\|\|(I - \pi_n)T(I - \pi_n)\varphi_n\|.\end{aligned} \tag{32}$$

Following (17),

$$(I - \pi_n)\varphi_n = (\pi_n - I)f, \tag{33}$$

so

$$\|\varphi - \varphi_n\| \leq \|R(z)\| \|(I - \pi_n)T\pi_n\varphi_n\| + (1 + \|\pi_n\|)\|R(z)\| \|T\| \|(I - \pi_n)f\|. \tag{34}$$

This estimate will be developed in the space $L^1([0, 1])$ in Section 4.

3. WORKING IN $X := C^0([0, 1])$

We recall that the oscillation of a function $x \in X$ relative to $\delta > 0$ is defined by

$$\omega(x, \delta) := \sup\{|x(\tau) - x(\tau')| : \tau, \tau' \in [0, 1], |\tau - \tau'| < \delta\}. \tag{35}$$

Let us consider the sequence of piecewise affine interpolatory projections in X: Let $(\tau_{n,j})_{j=1}^n$ be a grid on $[0, 1]$ such that

$$0 =: \tau_{n,1} < \tau_{n,2} < \cdots < \tau_{n,n-1} < \tau_{n,n} := 1. \tag{36}$$

Set

$$h_{n,j} := \tau_{n,j} - \tau_{n,j-1} \quad \text{for } j \in [\![2, n]\!]. \tag{37}$$

We define, for $x \in C^0([0, 1])$ and $j \in [\![1, n]\!]$,

$$\langle x, e_{n,j}^* \rangle := x(\tau_{n,j}) \tag{38}$$

and, for $\tau \in [0, 1]$ and $j \in [\![2, n-1]\!]$,

$$\begin{aligned} e_{n,j}(\tau) &:= \begin{cases} 1 + (\tau - \tau_{n,j})/h_{n,j} & \text{if } \tau \in [\tau_{n,j-1}, \tau_{n,j}], \\ 1 + (\tau_{n,j} - \tau)/h_{n,j+1} & \text{if } \tau \in [\tau_{n,j}, \tau_{n,j+1}], \\ 0 & \text{otherwise,} \end{cases} \\ e_{n,1}(\tau) &:= \begin{cases} (\tau_{n,2} - \tau)/h_{n,2} & \text{if } \tau \in [0, \tau_{n,2}], \\ 0 & \text{otherwise,} \end{cases} \\ e_{n,n}(\tau) &:= \begin{cases} (\tau - \tau_{n,n-1})/h_{n,n} & \text{if } \tau \in [\tau_{n,n-1}, 1], \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{39}$$

Clearly, $\langle e_{n,j}, e_{n,i}^* \rangle = \delta_{i,j}$ for $i, j \in [\![1, n]\!]$.

With these choices, the operator π_n is defined by (19) and produces the continuous piecewise affine interpolation on $[0, 1]$ relative to the grid $(\tau_{n,j})_{j=1}^n$.

Lemma 1. *The sequence $(\pi_n)_{n\geq 2}$ is bounded.*

Proof. $\|\pi_n\|_\infty = \sup\{\|\pi_n x\|_\infty : x \in C^0([0, 1]), \|x\|_\infty = 1\} = 1.$ ∎

Let

$$h_n := \max\{h_{n,j} : j \in [\![2, n]\!]\}. \tag{40}$$

Lemma 2. *For all $y \in X$,*

$$d_\infty(y, X_n) \leq \|(I - \pi_n)y\|_\infty \leq \omega(y, h_n), \tag{41}$$

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and hence, if $\lim_{n\to\infty} h_n = 0$, *then*

$$\lim_{n\to\infty} \|(I-\pi_n)x\|_\infty = 0$$

for each $x \in X$.

Proof. We have

$$\begin{aligned} d_\infty(y, X_n) &:= \inf_{x\in X_n} \|y-x\|_\infty \le \|(I-\pi_n)y\|_\infty \\ &= \max_{0\le\tau\le 1}\left|\sum_{j=1}^{n}(y(\tau)-y(t_{n,j}))e_{n,j}(\tau)\right| \le \omega(y,h_n), \end{aligned}$$

and the proof is complete. ■

We define, for $\delta > 0$,

$$\varepsilon(T,\delta) := \sup_{|\sigma-\tau|\le\delta}\int_0^1 |g(|\sigma-\tau'|)-g(|\tau-\tau'|)|\,d\tau'. \tag{42}$$

Lemma 3. *For all* $x \in X$ *and all* $\delta > 0$,

$$\omega(Tx,\delta) \le \varepsilon(T,\delta)\|x\|_\infty. \tag{43}$$

Proof. Evident. ■

Lemma 4. *For all* $\delta > 0$ *small enough*,

$$\varepsilon(T,\delta) \le 4\int_0^\delta g(t)\,dt. \tag{44}$$

Proof. For $\delta > 0$ and σ and τ in $[0,1]$ such that $\sigma < \tau$ and $|\sigma-\tau| \le \delta$, let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(t) := |g(|\sigma-t|)-g(|\tau-t|)|.$$

Set

$$r := \frac{\sigma+\tau}{2}.$$

It can be easily checked that the restriction of f to $[\sigma,\tau]$ is symmetric with respect to r and hence

$$\int_0^1 f(t)\,dt = \int_0^\sigma f(t)\,dt + 2\int_\sigma^r f(t)\,dt + \int_\tau^1 f(t)\,dt.$$

Let G be defined by (6). Then

$$\begin{aligned}\int_0^{\sigma} f(t)dt &= \int_0^{\sigma} (g(\sigma - t) - g(\tau - t))\, dt \\ &= G(\sigma) + G(\tau - \sigma) - G(\tau), \\ \int_{\sigma}^{r} f(t)\, dt &= \int_{\sigma}^{r} (g(t - \sigma) - g(\tau - t))\, dt \\ &= G((\tau - \sigma)/2) + G((\tau - \sigma)/2) - G(\tau - \sigma), \\ \int_{\tau}^{1} f(t)\, dt &= \int_{\tau}^{1} (g(t - \tau) - g(t - \sigma))\, dt \\ &= G(1 - \tau) - G(1 - \sigma) + G(\tau - \sigma).\end{aligned}$$

Hence

$$\begin{aligned}\varepsilon(T, \delta) &= 4\int_0^{(\tau-\sigma)/2} g(t)\, dt - \int_{\sigma}^{\tau} g(t)\, dt - \int_{1-\tau}^{1-\sigma} g(t)\, dt \le 4\int_0^{(\tau-\sigma)/2} g(t)\, dt \\ &\le 4\int_0^{\delta/2} g(\tau)\, d\tau \le 4\int_0^{\delta} g(\tau)\, d\tau,\end{aligned}$$

and the proof is complete. ■

Theorem 2. *Let $\varphi \ne 0$ be the solution of (8) with T defined by (1). Let φ_n be the solution of (10) with T_n defined by (9) and (36)–(39). Then*

$$\frac{\|\varphi_n - \varphi\|_\infty}{\|\varphi\|_\infty} \le 8c_0(z)\int_0^{h_n} g(\tau)\, d\tau. \tag{45}$$

Proof. It is a consequence of the preceding lemmas and of the estimate (42). ■

We remark that, in general, if the function f is symmetric with respect to the middle point of [0, 1], then the solution φ of (8) is symmetric with respect to the middle point of [0, 1]. In fact, the following relations are equivalent:

$$\int_0^1 g(|\tau - \tau'|)\, \varphi(\tau')\, d\tau' - z\varphi(\tau) = f(\tau) \tag{46}$$

$$-\int_1^0 g(|\tau - 1 + \tau'|)\, \varphi(1 - \tau')\, d\tau' - z\varphi(\tau) = f(\tau) \tag{47}$$

$$\int_0^1 g(|\tau' - \tau|)\, \varphi(1 - \tau')\, d\tau' - z\varphi(1 - \tau) = f(1 - \tau) \tag{48}$$

$$\int_0^1 g(|\tau' - \tau|)\psi(\tau')\, d\tau' = z\psi(\tau), \tag{49}$$

where

$$\psi(\tau) := \varphi(1 - \tau) - \varphi(\tau). \tag{50}$$

But since $T - zI$ is injective, then $\psi = 0$, that is φ is symmetric with respect to the middle point of $[0, 1]$.

4. WORKING IN $X := L^1([0, 1])$

Let $(\tau_{n,j})_{j=0}^{n}$ be a grid on $[0, 1]$ such that

$$0 =: \tau_{n,0} < \tau_{n,1} < \ldots < \tau_{n,n-1} < \tau_{n,n} := 1, \tag{51}$$

and set

$$h_{n,j} := \tau_{n,j} - \tau_{n,j-1} \quad \text{for } j \in [\![1, n]\!], \tag{52}$$

$$h_n := \max\{h_{n,j} : j \in [\![1, n]\!]\}. \tag{53}$$

We define, for $x \in L^1([0, 1])$,

$$\langle x, e_{n,j}^* \rangle := \frac{1}{h_{n,j}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} x(\tau')\, d\tau', \tag{54}$$

and, for $\tau \in [0, 1]$,

$$e_{n,j}(\tau) := \begin{cases} 1 & \text{if } \tau \in]\tau_{n,j-1}, \tau_{n,j}[, \\ 0 & \text{otherwise.} \end{cases} \tag{55}$$

It is clear that $\langle e_{n,j}, e_{n,i}^* \rangle = \delta_{i,j}$ for $i, j \in [\![1, n]\!]$.

Lemma 5. *If* $\lim_{n\to\infty} h_n = 0$, *then, for all* $x \in L^1([0, 1])$,

$$\lim_{n\to\infty} \|(I - \pi_n)x\|_1 = 0.$$

Proof. The convergence of $\pi_n x$ to x, for each $x \in L^1([0, 1])$ may be established by proving it for $x \in C^0([0, 1])$ (this will be done using the uniform continuity of x on $[0, 1]$ and using the Intermediate Value Theorem for integrals on each subinterval $]\tau_{n,j-1}, \tau_{n,j}[$, $j \in [\![1, n]\!]$). Next, use the density of $C^0([0, 1])$ in $L^1([0, 1])$ and apply Banach-Steinhaus theorem to the sequence $(\pi_n)_{n\geq 1}$ which is bounded:

$$\|\pi_n\|_1 = \sup\{\|\pi_n x\|_1 : x \in L^1([0, 1]), \|x\|_1 = 1\} = 1, \tag{56}$$

since

$$\begin{aligned} \|\pi_n x\|_1 &\leq \int_0^1 \sum_{j=1}^{n} \frac{1}{h_{n,j}} \left(\int_{\tau_{n,j-1}}^{\tau_{n,j}} |x(\tau')| d\tau' \right) e_{n,j}(\tau)\, d\tau \\ &= \sum_{j=1}^{n} \left(\int_{\tau_{n,j-1}}^{\tau_{n,j}} |x(\tau')| d\tau' \right) \left(\frac{1}{h_{n,j}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} d\tau \right) = \|x\|_1 \end{aligned} \tag{57}$$

and the bound is attained for $x \equiv 1$. ■

Let

$$\mu_n := \min\{h_{n,j} \colon j \in [\![1, n]\!]\}, \tag{58}$$

$$q_n := \frac{\mu_n}{h_n}. \tag{59}$$

The parameter q_n measures the uniformity of the grid: for *quasi-uniform* grids, there exists a constant q independent of n such that, for all n, $q \le q_n$, and for uniform grids $q_n = 1$ for all n.

Lemma 6. *If* $f \in C^0(]0,1]) \cap L^1([0,1])$ *is positive and decreasing on* $]0,1]$*, then*

$$\|(I - \pi_n) f\|_1 \le 2\left(1 + \frac{1}{q_n}\right) \int_0^{h_n} f(\tau)\, d\tau. \tag{60}$$

Proof. For $\tau \in]\tau_{n,i-1}, \tau_{n,i}[$,

$$(I - \pi_n) f(\tau) = f(\tau) - \frac{1}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} f(\tau)\, d\tau.$$

Since f is continuous and injective on $]\tau_{n,i-1}, \tau_{n,i}[$, there exists a unique point $\tilde{\tau}_{n,i} \in]\tau_{n,i-1}, \tau_{n,i}[$ such that

$$f(\tilde{\tau}_{n,i}) = \frac{1}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} f(\tau)\, d\tau,$$

$$f(\tau) > f(\tilde{\tau}_{n,i}) \quad \text{if } \tau_{n,i-1} < \tau < \tilde{\tau}_{n,i},$$

$$f(\tau) < f(\tilde{\tau}_{n,i}) \quad \text{if } \tilde{\tau}_{n,i} < \tau < \tau_{n,i}.$$

Hence

$$\begin{aligned}
\int_{\tau_{n,i-1}}^{\tau_{n,i}} |(I - \pi_n) f(\tau)| d\tau &= \int_{\tau_{n,i-1}}^{\tilde{\tau}_{n,i}} f(\tau)\, d\tau - \int_{\tilde{\tau}_{n,i}}^{\tau_{n,i}} f(\tau)\, d\tau \\
&\quad + f(\tilde{\tau}_{n,i})(\tau_{n,i} + \tau_{n,i-1} - 2\tilde{\tau}_{n,i}) \\
&= 2\int_0^{\tilde{\tau}_{n,i}} f(\tau)\, d\tau - \int_0^{\tau_{n,i}} f(\tau)\, d\tau - \int_0^{\tau_{n,i-1}} f(\tau)\, d\tau \\
&\quad + f(\tilde{\tau}_{n,i})(\tau_{n,i} + \tau_{n,i-1} - 2\tilde{\tau}_{n,i}) \\
&= 2\left[\int_{\tau_{n,i-1}}^{\tilde{\tau}_{n,i}} f(\tau)\, d\tau - f(\tilde{\tau}_{n,i})(\tilde{\tau}_{n,i} - \tau_{n,i-1})\right],
\end{aligned}$$

since

$$\int_0^{\tau_{n,i}} f(\tau)\, d\tau = \int_0^{\tau_{n,i-1}} f(\tau)\, d\tau + h_{n,i} f(\tilde{\tau}_{n,i}).$$

Thus

$$\|(I-\pi_n)f\|_1 = 2\Bigg[\int_0^{\tilde{\tau}_{n,1}} f(\tau)\,d\tau + \sum_{i=2}^{n}\int_{\tau_{n,i-1}}^{\tilde{\tau}_{n,i}} f(\tau)\,d\tau - \sum_{i=2}^{n} f(\tilde{\tau}_{n,i})(\tilde{\tau}_{n,i}-\tau_{n,i-1}) - f(\tilde{\tau}_{n,1})\tilde{\tau}_{n,1}\Bigg].$$

But f is a positive decreasing function, so

$$f(\tilde{\tau}_{n,i})(\tilde{\tau}_{n,i}-\tau_{n,i-1}) > f(\tau_{n,i})(\tilde{\tau}_{n,i}-\tau_{n,i-1}),$$
$$\int_{\tau_{n,i-1}}^{\tilde{\tau}_{n,i}} f(\tau)\,d\tau < f(\tau_{n,i-1})(\tilde{\tau}_{n,i}-\tau_{n,i-1}),$$

and hence

$$\begin{aligned}\|(I-\pi_n)f\|_1 &\le 2\Bigg[\int_0^{\tilde{\tau}_{n,1}} f(\tau)\,d\tau - f(\tilde{\tau}_{n,1})\tilde{\tau}_{n,1} + \sum_{i=2}^{n}(f(\tau_{n,i-1})-f(\tau_{n,i}))(\tilde{\tau}_{n,i}-\tau_{n,i-1})\Bigg]\\
&\le 2\Bigg[\int_0^{\tilde{\tau}_{n,1}} f(\tau)\,d\tau - f(\tilde{\tau}_{n,1})\tilde{\tau}_{n,1} + h_n(f(\tau_{n,1})-f(1))\Bigg]\\
&\le 2\Bigg[\int_0^{h_n} f(\tau)\,d\tau + h_n f(h_{n,1})\Bigg]\\
&\le 2\Bigg[\int_0^{h_n} f(\tau)\,d\tau + \frac{\mu_n}{q_n} f(\mu_n)\Bigg]\\
&\le 2\left(1+\frac{1}{q_n}\right)\int_0^{h_n} f(\tau)\,d\tau,\end{aligned}$$

which ends the proof. ■

A direct application of this lemma leads to

Theorem 3. *Let $f \in L^1([0,1])$ be such that there is a finite set of p abscissas $\{a_j : j \in [\![1,p]\!]\} \subset [0,1]$ verifying, $\cup_{j=2}^{p}[a_{j-1},a_j] = [0,1]$ and, for all $j \in]\!]2,p]\!]$, the restriction of f to $]a_{j-1},a_j[$ is continuous, of constant sign and monotone. Also, we suppose that the grid $(\tau_{n,j})_{j=0}^{n}$ contains the set $\{a_j : j \in [\![1,p]\!]\}$ for all n. Then*

$$\|(I-\pi_n)f\|_1 \le 2\left(1+\frac{1}{q_n}\right)\sum_{j=1}^{p}\int_{a_j-h_n}^{a_j+h_n} |f(\tau)|\,d\tau, \tag{61}$$

where f is extended by 0 outside the interval $[0,1]$.

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As a consequence, we have

Theorem 4. *Let $\varphi \neq 0$ be the solution of (8) with T defined by (1). Let φ_n be the solution of (10) with T_n defined by (9) and (51)–(55). Then, for n large enough,*

$$\frac{\|\varphi - \varphi_n\|_1}{\|\varphi\|_1} \leq 8.8 \frac{c_0(z)}{q_n}\left(1 + \frac{1}{q_n}\right) \int_0^{h_n} g(\tau)\, d\tau + c_1(z) \frac{\|(I - \pi_n) f\|_1}{\|f\|_1}, \tag{62}$$

with

$$c_1(z) := 2\|T\|_1 \kappa_1(T - zI), \tag{63}$$

where κ_1 denotes the condition number in the L^1 norm relative to inversion.

Proof. The bound (34) gives

$$\|\varphi - \varphi_n\|_1 \leq \|R(z)\|_1 \|(I - \pi_n) T \pi_n \varphi_n\|_1 + 2\|R(z)\|_1 \|T\|_1 \|f\|_1 \frac{\|(I - \pi_n) f\|_1}{\|f\|_1},$$

but

$$\|f\|_1 = \|(T - zI)\varphi\|_1 \leq \|T - zI\|_1 \|\varphi\|_1$$

so

$$\|\varphi - \varphi_n\|_1 \leq \|R(z)\|_1 \|(I - \pi_n) T \pi_n \varphi_n\|_1 + 2\|T\|_1 \kappa_1(T - zI) \|\varphi\|_1 \frac{\|(I - \pi_n) f\|_1}{\|f\|_1}.$$

It remains to proof that

$$\|(I - \pi_n) T \pi_n \varphi_n\|_1 \leq 8.8 \|\varphi\|_1 \frac{1}{q_n}\left(1 + \frac{1}{q_n}\right) \int_0^{h_n} g(\tau)\, d\tau.$$

Since $\lim_{n\to\infty} \|\varphi - \varphi_n\|_1 = 0$, for n large enough, $\|\varphi - \varphi_n\|_1 < 0.1\|\varphi\|_1$ and hence $|\|\varphi_n\|_1 - \|\varphi\|_1| < 0.1\|\varphi\|_1$
so

$$\|\varphi_n\|_1 \leq 1.1 \|\varphi\|_1$$

and

$$\|(I - \pi_n) T \pi_n \varphi_n\|_1 \leq 1.1\, \|\varphi\|_1\, \|(I - \pi_n) T \pi_n\|_1.$$

But, for all $x \in L^1([0,1])$, there exist n constants $c_{n,j}$, $j \in [\![1, n]\!]$, such that

$$\pi_n x = \sum_{j=1}^{n} c_{n,j} e_{n,j},$$

$$\sum_{j=1}^{n} |c_{n,j}| h_{n,j} = \|\pi_n x\|_1 \leq \|x\|_1.$$

Also,

$$
\begin{aligned}
&\|(I-\pi_n)T\pi_n x\|_1 \\
&\quad = \int_0^1 \Big| \int_0^1 g(|s-t|)(\pi_n x)(t)dt \\
&\qquad - \sum_{i=1}^{n} \frac{1}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} \int_0^1 g(|\tau - t|)x(t)\, dt\, d\tau e_{n,i}(s) \Big| ds \\
&\quad = \int_0^1 \Big| \sum_{j=1}^{n} c_{n,j} \Big[\int_{\tau_{n,j-1}}^{\tau_{n,j}} g(|s-t|)dt \\
&\qquad - \sum_{i=1}^{n} \frac{1}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} \int_{\tau_{n,j-1}}^{\tau_{n,j}} g(|\tau - t|)\, dt\, d\tau e_{n,i}(s) \Big] \Big| ds \\
&\quad \le \sum_{j=1}^{n} |c_{n,j}| \int_0^1 \Big| f_{n,j}(s) - \sum_{i=1}^{n} \frac{1}{h_{n,i}} \int_{\tau_{n,i-1}}^{\tau_{n,i}} f_{n,j}(\tau)\, d\tau e_{n,i}(s) \Big| ds,
\end{aligned}
$$

where

$$
f_{n,j}(s) := \int_{\tau_{n,j-1}}^{\tau_{n,j}} g(|s-t|)\, dt.
$$

In order to apply the preceding theorem, we need to study the properties of $f_{n,j}$. With G defined by (6), we can write

$$
f_{n,j}(s) = \begin{cases} G(\tau_{n,j} - s) - G(\tau_{n,j-1} - s) & \text{if } 0 \le s \le \tau_{n,j-1}, \\ G(\tau_{n,j} - s) + G(s - \tau_{n,j-1}) & \text{if } \tau_{n,j-1} < s < \tau_{n,j}, \\ G(s - \tau_{n,j-1}) - G(s - \tau_{n,j}) & \text{if } \tau_{n,j} \le s \le 1. \end{cases}
$$

This proves that the positive continuous function $f_{n,j}$ is an increasing function in $[0, \hat{\tau}_{n,j}]$ and a decreasing function in $[\hat{\tau}_{n,j}, 1]$, where

$$
\hat{\tau}_{n,j} := \frac{\tau_{n,j-1} + \tau_{n,j}}{2}.
$$

Hence, the maximum value of $f_{n,j}$ is

$$
\begin{aligned}
\max_{s \in [0,1]} f_{n,j}(s) = f_{n,j}(\hat{\tau}_{n,j}) &= 2(G((\tau_{n,j} - \tau_{n,j-1})/2)) \\
&= 2 \int_0^{h_{n,j/2}} g(t)\, dt \le 2 \int_0^{h_n} g(t)\, dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|(I-\pi_n)T\pi_n x\|_1 &\le 2\Big(1 + \frac{1}{q_n}\Big) \sum_{j=1}^{n} |c_{n,j}| \int_{\hat{\tau}_{n,j} - h_n}^{\hat{\tau}_{n,j+h_n}} f_{n,j}(s)\, ds \\
&\le 4\Big(1 + \frac{1}{q_n}\Big) \sum_{j=1}^{n} |c_{n,j}| h_n f_{n,j}(\hat{\tau}_{n,j})
\end{aligned}
$$

$$\leq \frac{4}{q_n}\left(1+\frac{1}{q_n}\right)\sum_{j=1}^{n}|c_{n,j}|h_{n,j}f_{n,j}(\hat{\tau}_{n,j})$$

$$\leq \frac{8}{q_n}\left(1+\frac{1}{q_n}\right)\|x\|_1\int_0^{h_n} g(t)\,dt,$$

and the proof is complete. ■

5. NUMERICAL EXAMPLES AND CONCLUSIONS

Throughout all this section, we shall work on the case

$$g(\tau) := \frac{\alpha}{\sqrt{\tau}}, \quad \tau \in]0,1], \text{ and } z = 1, \tag{64}$$

for different values of

$$\alpha \in \left]0, \frac{1}{2\sqrt{2}}\right[.$$

Since, both for $p = 1$ and for $p = \infty$,

$$\|T\|_p = 2\alpha\sqrt{2} < 1, \tag{65}$$

the value $z = 1$ belongs to the resolvent set of T. Moreover,

$$\|R(1)\|_p \leq \frac{1}{1-2\alpha\sqrt{2}}, \tag{66}$$

$$\|R_n(1)\|_p \leq \frac{1}{1-2\alpha\sqrt{2}}, \quad n \geq 1, \tag{67}$$

$$c_0(1) \leq \frac{1}{1-2\alpha\sqrt{2}}, \tag{68}$$

$$\|T - I\|_p \leq 1 + 2\alpha\sqrt{2}, \tag{69}$$

$$c_1(1) \leq 4\alpha\sqrt{2}\,\frac{1+2\alpha\sqrt{2}}{1-2\alpha\sqrt{2}}. \tag{70}$$

A continuous solution: For

$$f(\tau) := 2\alpha(\sqrt{\tau} + \sqrt{1-\tau}) - 1, \quad \tau \in [0,1], \tag{71}$$

the unique solution of (8) is

$$\varphi(\tau) := 1, \quad \tau \in [0,1]. \tag{72}$$

We consider $n = 500$ and a uniform grid. Depending on the value of α, we find the following a priori relative error estimates and computed relative errors:

α	A Priori Relative Error Estimate	Computed Relative Error
$(.2)10^{+0}$	$(.218)10^{+0}$	$(.647)10^{-04}$
$(.2)10^{-1}$	$(.148)10^{-1}$	$(.584)10^{-06}$
$(.2)10^{-2}$	$(.144)10^{-2}$	$(.630)10^{-08}$
$(.2)10^{-3}$	$(.144)10^{-3}$	$(.628)10^{-10}$

A bounded discontinuous integrable solution: For

$$f(\tau) := \begin{cases} 2\alpha(\sqrt{\tau} + \sqrt{1/2 - \tau}) - 1 & \text{if } \tau \in [0, 1/2], \\ 2\alpha(\sqrt{\tau} - \sqrt{\tau - 1/2}) & \text{if } \tau \in]1/2, 1], \end{cases} \tag{73}$$

the unique solution of (8) is the equivalence class in $L^1([0, 1])$ represented by

$$\varphi(\tau) := \begin{cases} 1 & \text{if } \tau \in [0, 1/2], \\ 0 & \text{if } \tau \in]1/2, 1], \end{cases} \tag{74}$$

We consider $n = 500$ and a uniform grid. Depending on the value of α, we find the following a priori relative error estimates and computed relative errors:

α	A Priori Relative Error Estimate	Computed Relative Error
$(.2)10^{+0}$	$(.644)10^{+0}$	$(.389)10^{-3}$
$(.2)10^{-1}$	$(.403)10^{-1}$	$(.389)10^{-4}$
$(.2)10^{-2}$	$(.389)10^{-2}$	$(.389)10^{-5}$
$(.2)10^{-3}$	$(.388)10^{-3}$	$(.389)10^{-6}$

We fix $\alpha = (.2)10^{-2}$. We consider two uniform grids: $\mathcal{G}_1$ for which $n = 500$, and $\mathcal{G}_2$ for which $n = 5000$. Next, two quasi-uniform grids are considered: $\mathcal{G}_3$ with $n = 650$, and $\mathcal{G}_4$ with $n = 1550$, as defined in the following table.

Grid	n	For j in	$h_{n,j}$	q_n
$\mathcal{G}_1$	500	$[\![1, 500]\!]$	$(.2)10^{-2}$	1
$\mathcal{G}_2$	5000	$[\![1, 5000]\!]$	$(.2)10^{-3}$	1
$\mathcal{G}_3$	650	$[\![1, 100]\!] \cup [\![251, 450]\!]$	$(.1)10^{-2}$	$\frac{1}{2}$
		$[\![101, 250]\!] \cup [\![451, 650]\!]$	$(.2)10^{-2}$	
$\mathcal{G}_4$	1550	$[\![1, 400]\!] \cup [\![551, 1350]\!]$	$(.25)10^{-3}$	$\frac{1}{8}$
		$[\![401, 550]\!] \cup [\![1351, 1550]\!]$	$(.2)10^{-2}$	

Here below, we compare a priori relative error estimates and computed relative errors for the grids considered before.

Grid	A Priori Relative Error Estimate	Computed Relative Error
$\mathcal{G}_1$	$(.389)10^{-2}$	$(.389)10^{-5}$
$\mathcal{G}_2$	$(.107)10^{-2}$	$(.394)10^{-6}$
$\mathcal{G}_3$	$(.106)10^{-1}$	$(.250)10^{-5}$
$\mathcal{G}_4$	$(.117)10^{+0}$	$(.144)10^{-5}$

As expected, the theoretical error estimates become less realistic as q_n decreases, that is, as the grid looses uniformity. We observe that the true relative error is divided by 10 when passing from $\mathcal{G}_1$ to $\mathcal{G}_2$, that is when the number of nodes is multiplied by 10 keeping a uniform grid.

An unbounded integrable solution: For

$$f(\tau) := \alpha\pi - \frac{1}{\sqrt{|\tau - 1/2|}} + \alpha \ln\left(\frac{3 - 2\tau + 2\sqrt{2(1-\tau)}}{2\tau + 1 - 2\sqrt{2\tau}}\right), \tag{75}$$

where $\tau \in [0, 1]\backslash\{1/2\}$, the unique solution of (8) is the equivalence class in $L^1([0, 1])$ represented by

$$\varphi(\tau) := \frac{1}{\sqrt{|\tau - 1/2|}}, \quad \tau \in [0, 1]\backslash\{1/2\}. \tag{76}$$

We consider $n = 500$ and a uniform grid. Depending on the value of α, we find the following a priori relative error estimates and computed relative errors:

α	A Priori Relative Error Estimate	Computed Relative Error
$(.2)10^{+0}$	$(.651)10^{-1}$	$(.364)10^{-1}$
$(.2)10^{-1}$	$(.606)10^{-2}$	$(.328)10^{-2}$
$(.2)10^{-2}$	$(.602)10^{-3}$	$(.325)10^{-3}$
$(.2)10^{-3}$	$(.602)10^{-4}$	$(.325)10^{-4}$

We remark that the theoretical error estimate is quite realistic in this case in which the solution has a vertical asymptote.

Let us fix $\alpha = (.2)10^{-2}$. First we consider the uniform grid $\mathcal{G}_1$ for which $n = 500$. Next, two quasi-uniform grids are considered: $\mathcal{G}_5$ and $\mathcal{G}_6$, as defined in the following table.

Grid	n	for j in	$h_{n,j}$	q_n
$\mathcal{G}_1$	500	$[\![1, 500]\!]$	$(.20)10^{-2}$	1
$\mathcal{G}_5$	600	$[\![1, 200]\!] \cup [\![401, 600]\!]$	$(.20)10^{-2}$	$\frac{1}{2}$
		$[\![201, 400]\!]$	$(.10)10^{-2}$	
$\mathcal{G}_6$	1200	$[\![1, 200]\!] \cup [\![1001, 1200]\!]$	$(.20)10^{-2}$	$\frac{1}{8}$
		$[\![201, 1000]\!]$	$(.25)10^{-3}$	

Here below, we compare a priori relative error estimates and computed relative errors for the grids considered before.

Grid	A Priori Relative Error Estimate	Computed Relative Error
$\mathcal{G}_1$	$(.606)10^{-2}$	$(.328)10^{-2}$
$\mathcal{G}_5$	$(.139)10^{-1}$	$(.328)10^{-2}$
$\mathcal{G}_6$	$(.127)10^{+0}$	$(.328)10^{-2}$

As in the previous application, the theoretical error estimates become less realistic as q_n decreases, that is, as the grid looses uniformity.

About the figures: Figures which follow show the function f and the corresponding numerical solution for a prescribed value of α, $n = 500$ and a uniform grid; local behaviors are enhanced in Figures 5 and 6, and local comparisons with the exact solution are shown in Figures 9 and 10. Figure 11 shows the relative error function $(\varphi_n - \varphi)/\|\varphi\|_1$ corresponding to the nonuniform grid $\mathcal{G}_6$. Figure 12 enhances a local comparison between two approximate solutions and the exact one in the unbounded case.

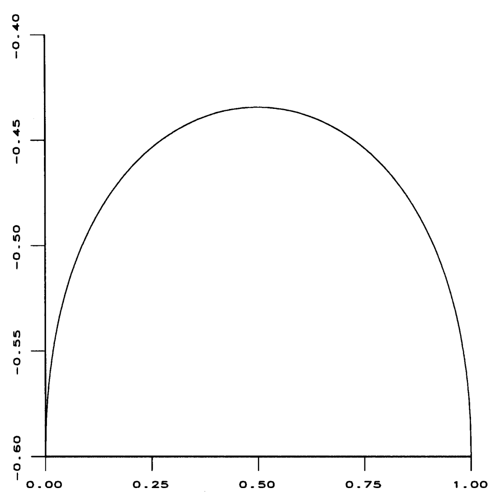

Figure 1. $f(\tau) := 0.4(\sqrt{\tau} + \sqrt{1-\tau}) - 1$, $\alpha = 0.2$.

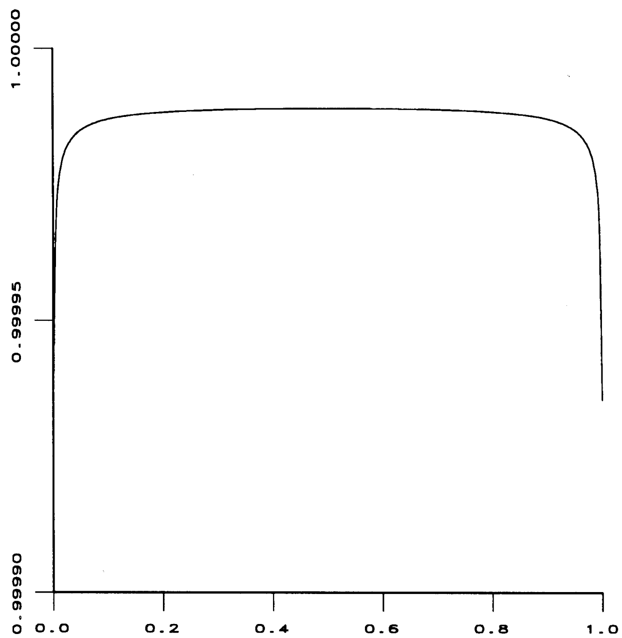

Figure 2. Approximate solution with $n = 500$ and equally spaced nodes corresponding to the function f given in Figure 1.

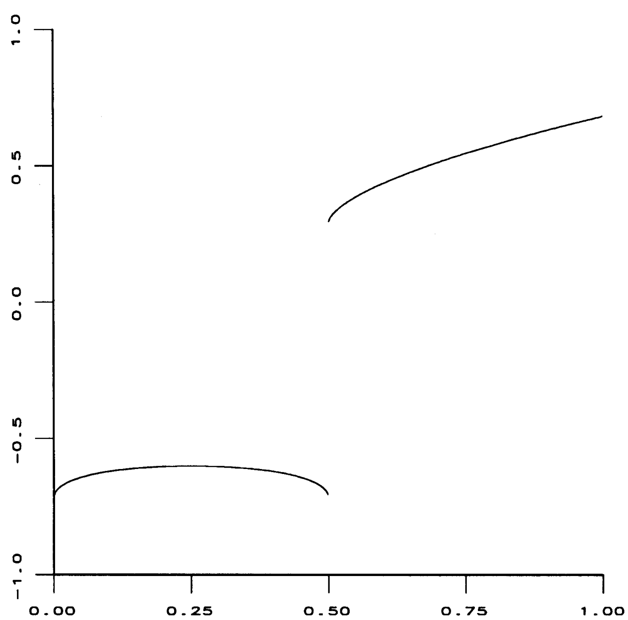

Figure 3. $f(\tau) := 0.4(\sqrt{\tau} + \sqrt{1/2 - \tau}) - 1$ if $\tau \in [0, 1/2]$, and $f(\tau) := 0.4(\sqrt{\tau} - \sqrt{\tau - 1/2})$ if $\tau \in]1/2, 1]$, $\alpha = 0.2$.

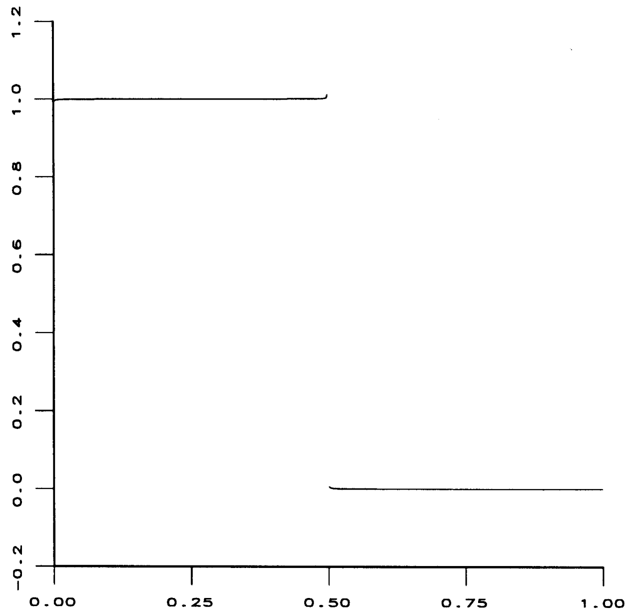

Figure 4. Approximate solution with $n = 500$ and equally spaced nodes corresponding to the function f given in Figure 3.

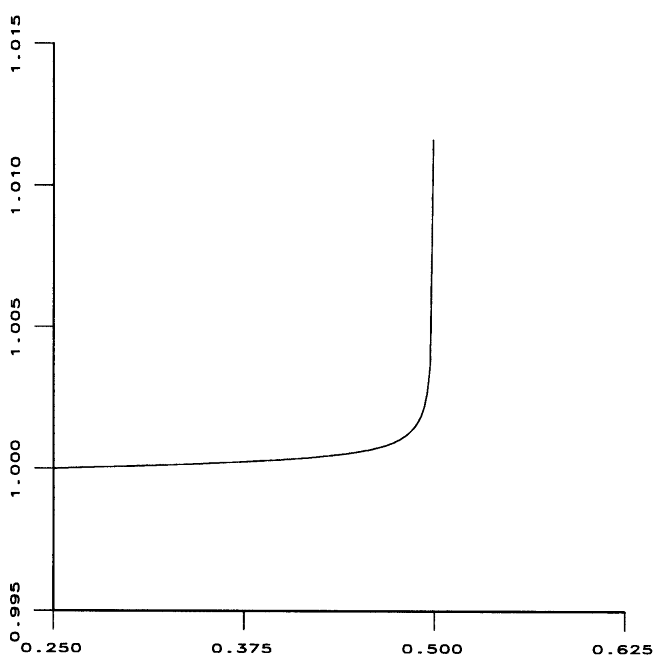

Figure 5. Zoom of Figure 4 around $\tau = 0.4$.

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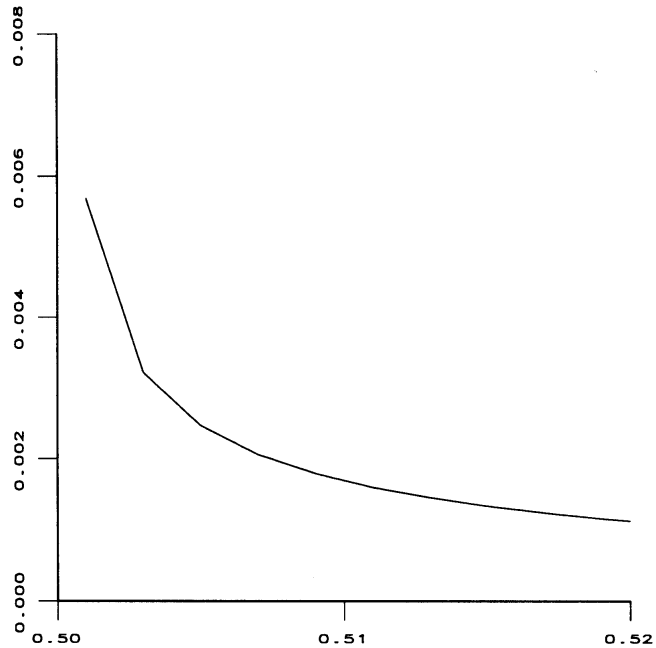

Figure 6. Zoom of Figure 4 around $\tau = 0.51$.

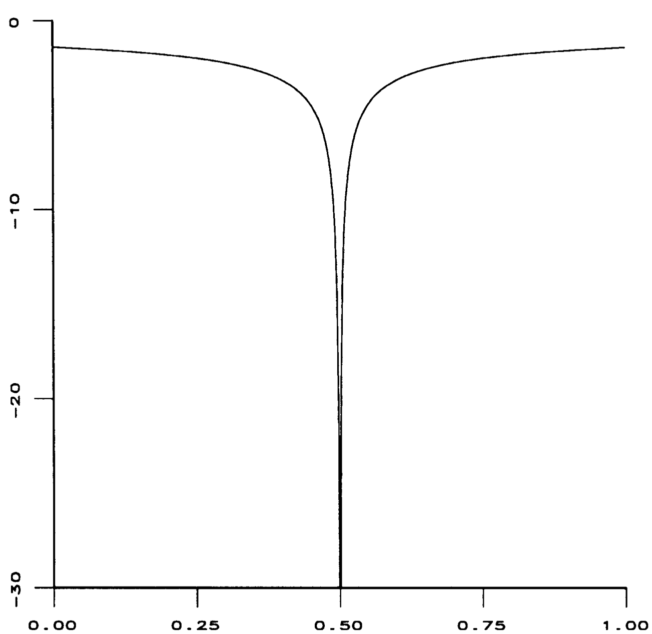

Figure 7. $f(\tau) := 0.002\pi - (1/\sqrt{|\tau - 1/2|}) + 0.002\ln((3 - 2\tau + 2\sqrt{2(1-\tau)})/(2\tau + 1 - 2\sqrt{2\tau}))$, $\alpha = 0.002$.

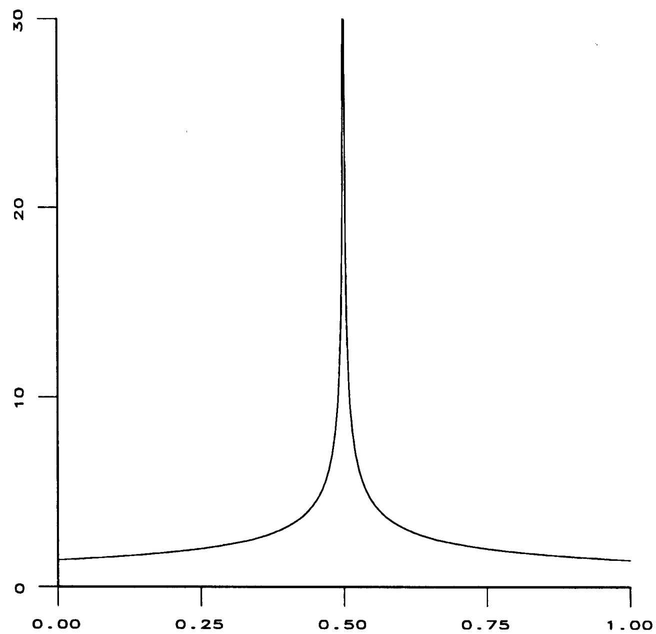

Figure 8. Approximate solution with $n = 500$ and equally spaced nodes corresponding to the function f given in Figure 7.

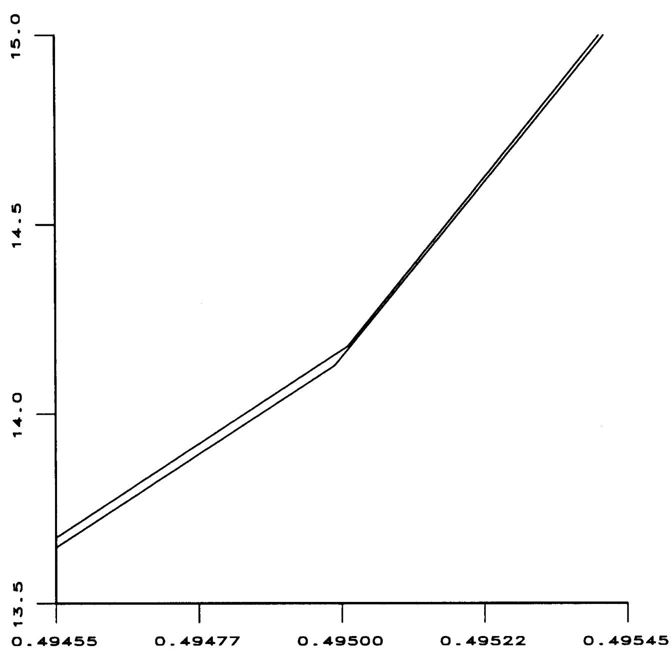

Figure 9. Zoom of Figure 8 and the exact solution around $\tau = 0.495$.

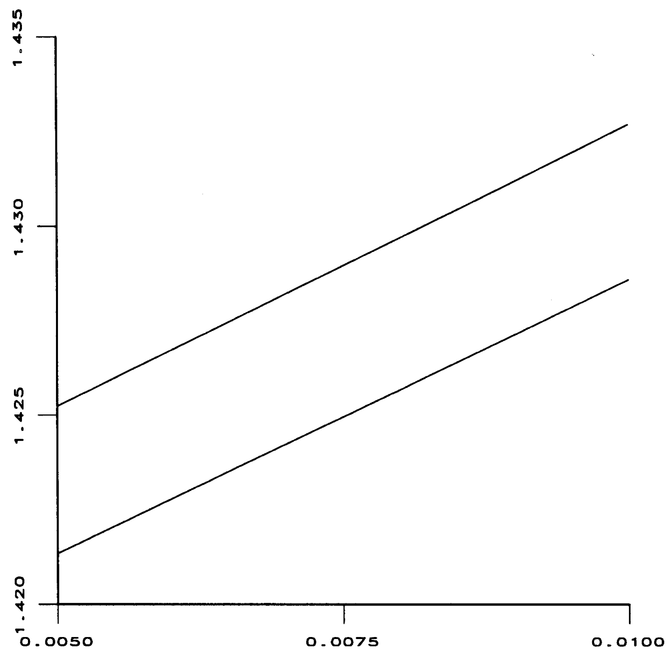

Figure 10. Zoom of Figure 8 and the exact solution around $\tau = 0.0075$.

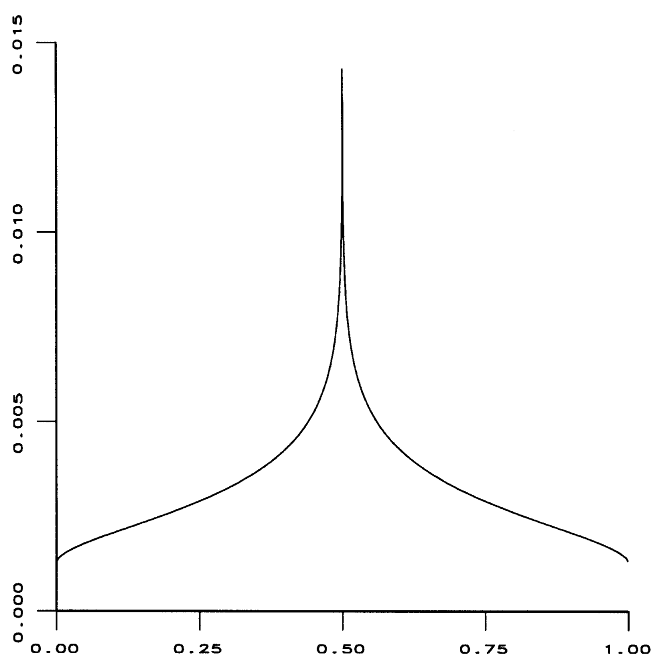

Figure 11. Relative error function corresponding to $\mathcal{G}_6$ in the unbounded integrable case with $\alpha = (.2)10^{-2}$.

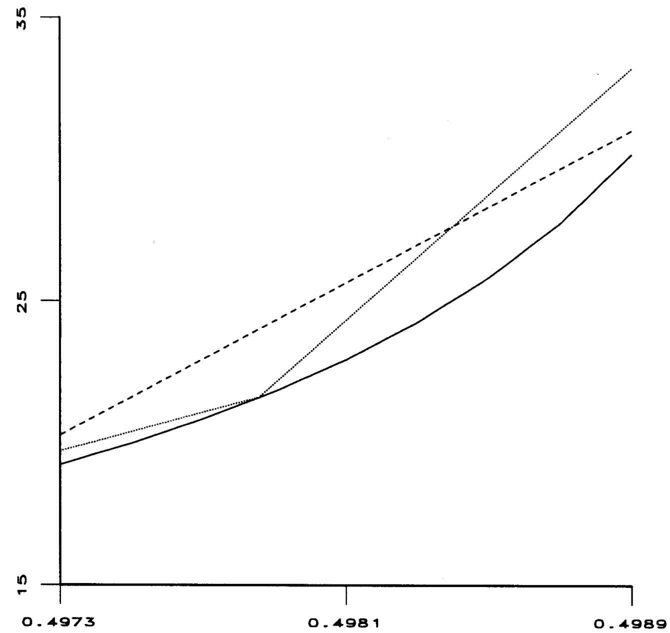

Figure 12. Local comparison of two uniform grid approximate solutions φ_{500} with dashes, φ_{700} with dots, and the exact solution with a line, in the unbounded integrable case with $\alpha = (.2)10^{-2}$.

Final remarks and bibliographical comments: In this paper we have dealt with projection approximations of Fredholm integral operators. Other types of approximations which apply to weakly singular operators are developed in [10] and [11], where collocation methods are viewed as projection type discretizations, and [6], [3] and [8] where singularity subtraction techniques are presented.

In the case of the space $L^1([0,1])$, we can build another relative error estimate which does not depend on the free term f by introducing the L^1-oscillation of a Lebesgue integrable function x on $[0,1]$ relative to $\delta > 0$:

$$\omega_1(x,\delta) := \sup\left\{\int_0^1 |x(\tau+\tau') - x(\tau)|\, d\tau' \colon \tau \in [0,\delta]\right\}, \tag{77}$$

where x is extented by zero in the exterior of $[0,1]$. Indeed, we can prove that

$$\|(I-\pi_n)x\|_1 \le \frac{2}{q_n}\omega_1(x,h_n), \tag{78}$$

and that

$$\omega_1(Tx,h_n) \le \varepsilon(T,h_n)\|x\|_1. \tag{79}$$

Hence

$$\frac{\|\varphi - \varphi_n\|_1}{\|\varphi\|_1} \leq \frac{8}{q_n} \int_0^{h_n} g(\tau)\, d\tau. \tag{80}$$

The approximate operators studied in this work are norm convergent to the limit given operator. In general weaker conditions on the approximating sequence are enough to guaranty the convergence of the approximate solution to the exact one. A comprehensive treatment of the collectively compact convergence has been done by Anselone in [7] and further extensions can be found in [9]; as well, the bases of ν-convergent approximations are developed in [1], [2] and [4]. The reader interested in an application to the radiation transfer equation in stellar atmospheres is refered to [12] where the authors apply the methods of this paper to the convolution type integral operator defined through the weakly singular kernel induced by $g := E_1$, the first exponential-integral function:

$$\tau \mapsto E_1(\tau) := \frac{\varpi}{2} \int_0^1 \frac{e^{-\tau/\mu}}{\mu}\, d\mu, \quad \tau > 0,$$

and where an iterative refinement scheme is used to improve on the accuracy of the approximate solution both in sequential and parallel computation.

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